LECTURE NOTES ON MICROECONOMICS

ANALYZING MARKETS WITH BASIC CALCULUS

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Part 3: Firms and competition

Chapter 9: Production

Problems

> Note: Terminology varies for the |slope| of an isoquant. In my slideshows, I call it the "marginal rate of substitution in production" (MRSP). Hicks calls it simply the "marginal rate of substitution."[1] Textbooks often call it the "marginal rate of technical substitution."[2]

(9.1) [Production] Prove that the elasticity of output q with respect to input x necessarily equals the ratio of marginal product MP to average product.

(9.2) [Input substitution, returns to scale] Let q denote output, x_1 denote the number of machines, and x_2 denote the number of workers in a production process. Suppose output is produced according to the following function: $q = 5\,x_1 + 3x_2 - 7$.

a. Find a formula for the marginal product of machines. Are there diminishing returns to machines? Why or why not? [Hint: Does the marginal product decrease as x_1 increases and x_2 is held constant?]
b. Find a formula for the marginal product of workers. Are there diminishing returns to workers? Why or why not? [Hint: Does the marginal product decrease as x_2 increases and x_1 is held constant?]
c. Find a formula for the marginal rate of substitution in production of workers for machines, MRSP. [Hint: This is the |slope| of an isoquant when machines are on the vertical axis and workers are on the horizontal axis.]
d. Does this function have diminishing MRSP? Why or why not? [Hint: What happens to the MRSP as x_1 decreases and x_2 increases?]
e. Does this production function enjoy increasing, decreasing, or constant returns to scale? Why?

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[1] J. R. Hicks, *Value and Capital*, second edition, Oxford: Clarendon Press, 1978, p. 86.

[2] David A. Besanko and Ronald R. Braeutigam, *Microeconomics*, second edition, John Wiley & Sons, 2005, p. 198. Steven E Landsburg, *Price Theory and Applications*, fifth edition, Thomson South-Western, 2002, p. 160. Jeffrey M. Perloff, *Microeconomics: Theory and Applications with Calculus,"* Pearson Addison Wesley, 2008.

(9.3) [Input substitution, returns to scale] Let q denote output, x_1 denote the number of machines, and x_2 denote the number of workers in a production process. Suppose output is produced according to the following function: $q = 8\, x_1^{1/2}\, x_2^{3/4}$.

a. Find a formula for the marginal product of machines. Are there diminishing returns to machines? Why or why not? [Hint: Does the marginal product decrease as x_1 increases and x_2 is held constant?]
b. Find a formula for the marginal product of workers. Are there diminishing returns to workers? Why or why not? [Hint: Does the marginal product decrease as x_2 increases and x_1 is held constant?]
c. Find a formula for the elasticity of output with respect to machines.
d. Find a formula for the elasticity of output with respect to workers.
e. Find a formula for the marginal rate of substitution in production of workers for machines, MRSP.
f. Does this function have diminishing MRSP? Why or why not? [Hint: What happens to the MRSP as x_1 decreases and x_2 increases?]
g. Does this production function enjoy increasing, decreasing, or constant returns to scale? Justify your answer. If this production function characterized a real industry (like steel, restaurants, airlines, etc.) would you expect that industry to be dominated by very large firms, very small firms, or a mix of firms of all sizes?

(9.4) [Input substitution, returns to scale] Let q denote output, x_1 denote the number of machines, and x_2 denote the number of workers in a production process. Suppose output is produced according to the following function: $q = 5x_1 + 7x_2 + 4(x_1x_2)^{1/2}$.

a. Find a formula for the marginal product of machines. Are there diminishing returns to machines? Why or why not? [Hint: Does the marginal product decrease as x_1 increases and x_2 is held constant?]
b. Find a formula for the marginal product of workers. Are there diminishing returns to workers? Why or why not? [Hint: Does the marginal product decrease as x_2 increases and x_1 is held constant?]
c. Find a formula for the marginal rate of substitution in production of workers for machines, MRSP. [Hint: It is a little messy! Don't bother to try to simplify it.]
d. Does this function have diminishing MRSP? Why or why not? [Hint: What happens to the numerator of MRSP as x_1 decreases and x_2 increases? What happens to the denominator?]
e. Does this production function enjoy increasing, decreasing, or constant returns to scale? Justify your answer. If this production function characterized a real industry (like steel, restaurants, airlines, etc.) would you expect that industry to be dominated by very large firms, very small firms, or a mix of firms of all sizes?

(9.5) [Input substitution, returns to scale] Let q denote output, x_1 denote the number of machines, and x_2 denote the number of workers in a production process. Suppose output is produced according to the following function: $q = 2x_1^{1/2} + 3x_2^{1/2}$.

a. Find a formula for the marginal product of machines. Are there diminishing returns to machines? Why or why not? [Hint: Does the marginal product decrease as x_1 increases and x_2 is held constant?]
b. Find a formula for the marginal product of workers. Are there diminishing returns to workers? Why or why not? [Hint: Does the marginal product decrease as x_2 increases and x_1 is held constant?]
c. Find a formula for the marginal rate of substitution in production of workers for machines, MRSP.
d. Does this function have diminishing MRSP? Why or why not? [Hint: What happens to the numerator of MRSP as x_1 decreases and x_2 increases? What happens to the denominator?]
e. Does this production function enjoy increasing, decreasing, or constant returns to scale? Justify your answer. If this production function characterized a real industry (like steel, restaurants, airlines, etc.) would you expect that industry to be dominated by very large firms, very small firms, or a mix of firms of all sizes?

(9.6) [Input substitution, returns to scale] Let q denote output, x_1 denote the number of machines, and x_2 denote the number of workers in a production process. Suppose output is produced according to the following function: $q = (x_1^{-1} + x_2^{-1})^{-1}$.

a. Find a formula for the marginal product of machines. Are there diminishing returns to machines? Why or why not? [Hint: Does the marginal product decrease as x_1 increases and x_2 is held constant?]
b. Find a formula for the marginal product of workers. Are there diminishing returns to workers? Why or why not? [Hint: Does the marginal product decrease as x_2 increases and x_1 is held constant?]
c. Find a formula for the marginal rate of substitution in production of workers for machines, MRSP.
d. Does this function have diminishing MRSP? Why or why not? [Hint: What happens to the numerator of MRSP as x_1 decreases and x_2 increases? What happens to the denominator?]
e. Does this production function enjoy increasing, decreasing, or constant returns to scale? Justify your answer. If this production function characterized a real industry (like steel, restaurants, airlines, etc.) would you expect that industry to be dominated by very large firms, very small firms, or a mix of firms of all sizes?

(9.7) [Fixed-proportions production technology] Suppose a particular machine can make 50 parts per hour if it is operated by four workers. The machine cannot be operated by less than four workers, and extra workers on the same machine add nothing to output. A firm can use as many machines as desired, with no loss in output per machine, provided each machine is operated by at least four workers. Let x_1 denote the number of machines and x_2 denote the number of workers used by the firm.

a. Give an equation for the firm's so-called "expansion path"—that is, the efficient relationship between x_1 and x_2.
b. Sketch the firm's isoquants or describe them in words. Are the firm's isoquants curved, straight, or L-shaped?

Let q denote the number of parts made per hour. For simplicity, assume the number of machines x_1 is an integer and the number of workers x_2 is a multiple of four: $x_2 = 0, 4, 8, 12$, etc.

c. If there are plenty of workers, what is the relationship between output q and the number of machines x_1 ? Give an equation "q = ... "
d. If there are plenty of machines, what is the relationship between output q and the number of workers x_2 ? Give an equation "q = ... "
e. Give an equation for the firm's production function using the minimum function "*min*{ , } ". [Hint: The value of the minimum function by definition equals the smallest item inside the braces. Examples: $min\{2,9\} = 2$. $min\{10,8\} = 8$.]

(9.8) [Fixed-proportions production technology] Suppose a particular machine can make 100 parts per hour if it is operated by five workers. The machine cannot be operated by less than five workers, and extra workers on the same machine add nothing to output. A firm can use as many machines as desired, with no loss in output per machine, provided each machine is operated by at least five workers. Let x_1 denote the number of machines and x_2 denote the number of workers used by the firm.

a. Give an equation for the firm's so-called "expansion path"—that is, the efficient relationship between x_1 and x_2.
b. Sketch the firm's isoquants or describe them in words. Are the firm's isoquants curved, straight, or L-shaped?

Let q denote the number of parts made per hour. For simplicity, assume the number of machines x_1 is an integer and the number of workers x_2 is a multiple of five: $x_2 = 0, 5, 10, 15$, etc.

c. If there are plenty of workers, what is the relationship between output q and the number of machines x_1 ? Give an equation "q = ... "
d. If there are plenty of machines, what is the relationship between output q and the number of workers x_2 ? Give an equation "q = ... "
e. Give an equation for the firm's production function using the minimum function "*min*{ , } ". [Hint: The value of the minimum function by definition equals the smallest item inside the braces. Examples: $min\{2,9\} = 2$. $min\{10,8\} = 8$.]

The next questions refer to the following table of growth rates for the United States.

	Labor	Real capital	Real GDP
1951-1960	1.03%	2.63%	3.05%
1960-1970	1.81%	3.87%	4.20%
1970-1980	2.36%	5.04%	3.18%
1980-1990	1.81%	1.39%	3.24%
1990-2000	1.43%	2.95%	3.40%
2000-2010	0.16%	2.82%	1.54%

SOURCES: Labor: "Employment" (series ID LNU02000000) from www.bls.gov, downloaded August 2, 2012. Real capital: "Produced Assets Closing Balance" (table 5.9) divided by GDP price index (table 1.1.4) from www.bea.gov, downloaded August 2, 2012. Real GDP: "Real Gross Domestic Product" (table 1.1.6) from www.bea.gov, downloaded August 2, 2012.

(9.9) [Technical change] In the aggregate production function for the United States, the elasticity of output with respect to labor is about 0.7 and the elasticity of output with respect to capital is about 0.3. Analyze the data in the table above for the period 1951-1960.

a. If there were no technical change, by what percent should output have increased?
b. What is the Solow residual? That is, by how much did output increase solely due to technical change?

For comparison, analyze the data for the period 1980-1990.

c. If there were no technical change, by what percent should output have increased?
d. What is the Solow residual? That is, by how much did output increase solely due to technical change?

(9.10) [Technical change] In the aggregate production function for the United States, the elasticity of output with respect to labor is about 0.7 and the elasticity of output with respect to capital is about 0.3. Analyze the data in the table above for the period 1960-1970.

a. If there were no technical change, by what percent should output have increased?
b. What is the Solow residual? That is, by how much did output increase solely due to technical change?

For comparison, analyze the data for the period 1990-2000.

c. If there were no technical change, by what percent should output have increased?
d. What is the Solow residual? That is, by how much did output increase solely due to technical change?

(9.11) [Technical change] In the aggregate production function for the United States, the elasticity of output with respect to labor is about 0.7 and the elasticity of output with respect to capital is about 0.3. Analyze the data in the table above for the period 1970-1980.

a. If there were no technical change, by what percent should output have increased?
b. What is the Solow residual? That is, by how much did output increase solely due to technical change?

For comparison, analyze the data for the period 2000-2010.

c. If there were no technical change, by what percent should output have increased?
d. What is the Solow residual? That is, by how much did output increase solely due to technical change?

[end of problem set]