LECTURE NOTES ON MICROECONOMICS

ANALYZING MARKETS WITH BASIC CALCULUS

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Part 2: Consumers and demand

Chapter 5: Demand functions

Section 5.1: What is an individual demand function?

Determinants of choice. As noted in the previous chapter, we assume each consumer chooses the most preferred bundle that she or he can afford. Which bundle is chosen therefore depends on (1) preferences, as shown by indifference curves or a utility function, and (2) prices and income, as shown by the budget line. Note that any change in preferences, price or income will, in general, cause the consumer to make a new choice. In this chapter, we assume that preferences do not change. We focus on changes in consumer choice that are driven by changes in the budget line—that is, changes in prices and income.

Definition of demand function. If preferences do not change, then the quantities of each good chosen by a consumer depend on prices and income. For example, suppose there are only two goods. Denote the quantities of these goods q_1 and q_2. Denote their prices p_1 and p_2. Denote the consumer's income I. Then the quantities chosen by the consumer can be written as *demand functions.*

(5.1) $$q_1{}^* = q_1{}^*(p_1, p_2, I) \qquad \text{and} \qquad q_2{}^* = q_2{}^*(p_1, p_2, I)$$

Here, the asterisks indicate that these are not arbitrary quantities—these are the quantities of the consumer's most preferred affordable bundle. Put differently, they are the consumer's *utility-maximizing choices*, given the budget constraint. Note that these quantities are, in general, functions of all prices and income. (For some special preferences, some of the arguments may drop out of the demand function, as we will see later.) Prices and income are *taken as given* by the consumer. By contrast, the quantities of other goods are *not* arguments of the demand functions, because other quantities are *choices* made by the consumer

Section 5.2: Overall properties of legitimate demand functions

Note that for each good there must be a corresponding demand function. A collection of functions that define the quantities of all goods chosen in the consumer's bundle is called a *demand system.* However, not all purported demand functions or demand systems can be derived from utility-maximizing behavior by consumers. Certain properties must hold for these purported demand functions to be legitimate.

Budget constraint. Recall from the previous chapter that the consumer's choice will always lie on the budget line. It follows that total spending implied by a demand system must add up to the consumer's income. This is sometimes called the "adding-up property." For the two-good case, a demand system that satisfies the budget line must satisfy the following equation.[1]

(5.2) $$I = q_1 p_1 + q_2 p_2 = q_1{*}(p_1, p_2, I)\cdot p_1 + q_2{*}(p_1, p_2, I)\cdot p_2$$

Example: Assume that there are only two goods. Do the following functions satisfy the budget constraint?

(5.3) $$q_1{*} = \frac{I}{2p_1} + \frac{3}{p_1} + \frac{5p_2}{p_1} \qquad \text{and} \qquad q_2{*} = \frac{I}{2p_2} - \frac{3}{p_2} - 5$$

To answer this question, we must substitute the above formulas for q_1* and q_2* into equation (5.2) and check to see if the equation balances:

(5.4) $$\begin{aligned} I &\overset{?}{=} q_1{*}\cdot p_1 \quad + \quad q_2{*}\cdot p_2 \\ &= \left(\frac{I}{2p_1} + \frac{3}{p_1} + \frac{5p_2}{p_1}\right) p_1 \quad + \quad \left(\frac{I}{2p_2} - \frac{3}{p_2} - 5\right)\cdot p_2 \\ &= \left(\frac{I}{2} + 3 + 5p_2\right) \quad + \quad \left(\frac{I}{2} - 3 - 5p_2\right) \\ &= I \end{aligned}$$

The equation does indeed balance, so spending necessarily equals income and the budget constraint is indeed satisfied.

Homogeneity. Another property of demand functions is *homogeneity of degree zero* in all arguments. A function that is homogeneous of degree zero is defined as one whose value does not change if all its arguments change proportionally.[2] Why must demand functions be homogeneous of degree zero? Recall that if income and all prices rise as the same rate then the budget line does not change. If the budget line does not change, the consumer's most preferred bundle will certainly not change. Therefore demand functions must be homogeneous of degree zero. Formally, for any arbitrary positive number α, the following equations must hold:

(5.5) $$\begin{aligned} q_1{*}(\alpha p_1, \alpha p_2, \alpha I) &= q_1{*}(p_1, p_2, I) \\ q_2{*}(\alpha p_1, \alpha p_2, \alpha I) &= q_2{*}(p_1, p_2, I) \end{aligned}.$$

Homogeneity means that pure inflation will have no effect on a consumer's choices. Pure inflation is defined as a proportionate increase in income and all prices. Of course, pure

[1] Additional properties follow directly from the budget constraint. If equation (5.2) is differentiated with the respect to income I, the so-called Engel aggregation condition is derived. If the same equation is differentiated with respect to one price, the so-called Cournot aggregation condition is derived.

[2] More generally, a function $f(x_1,x_2,\ldots,x_n)$ is said to be *homogeneous of degree n* if, for any number α, $f(\alpha x_1,\alpha x_2,\ldots,\alpha x_n) = \alpha^n f(x_1,x_2,\ldots,x_n)$.

inflation is unusual in the real world. Actual experiences of inflation may cause some components of income to fall relative to others (such as retirement benefits) and some prices to fall relative to others.

Example: Is the function $q_1* = I/(3p_1) + (p_2/p_1)$ homogeneous of degree zero? Replacing I by αI, p_1 by αp_1, and p_2 by αp_2, and canceling yields the original demand function. So this demand function is indeed homogeneous of degree zero.

Example: Return to the functions used in a previous example:

$$q_1* = \frac{I}{2p_1} + \frac{3}{p_1} + \frac{5p_2}{p_1} \qquad \text{and} \qquad q_2* = \frac{I}{2p_2} - \frac{3}{p_2} - 5$$

Are these functions homogeneous of degree zero? To check this, we must multiply all arguments by an arbitrary number—call it α—and check to see if the quantity chosen is unchanged, as required by equation (5.5) above. Replacing I by αI, p_1 by αp_1, and p_2 by αp_2 yields the following.

$$\begin{aligned} q_1* &\overset{?}{=} \frac{\alpha I}{2\alpha p_1} + \frac{3}{\alpha p_1} + \frac{5\alpha p_2}{\alpha p_1} &&= \frac{I}{2p_1} + \frac{3}{\alpha p_1} + \frac{5p_2}{p_1} \\ q_2* &\overset{?}{=} \frac{\alpha I}{2\alpha p_2} - \frac{3}{\alpha p_1} - 5 &&= \frac{I}{2p_2} - \frac{3}{\alpha p_1} - 5 \end{aligned} \tag{5.6}$$

Though many of the αs in this example cancel, not all do so. Therefore, these functions do *not* satisfy homogeneity of degree zero and therefore cannot be legitimate individual demand functions.

Often Cobb-Douglas or constant-elasticity functions are used to model demand. These functions take the form $q_1* = b\, I^c\, p_1^d\, p_2^e$, where b, c, d, and e are constants. Replacing I by (αI), p_1 by (αp_1), and p_2 by (αp_2) yields the following:

$$\begin{aligned} q_1* &\overset{?}{=} b\,(\alpha I)^c\,(\alpha p_1)^d\,(\alpha p_2)^e \\ &= b\left(\alpha^c I^c\right)\left(\alpha^d p_1^d\right)\left(\alpha^e p_2^e\right) \\ &= b\, I^c\, p_1^d\, p_2^e\, \alpha^c\, \alpha^d\, \alpha^e \\ &= b\, I^c\, p_1^d\, p_2^e\, \alpha^{c+d+e} \end{aligned} \tag{5.7}$$

Thus the αs cancel if and only if $(c+d+e) = 0$, for only then will α^{c+d+e} equal one. So constant-elasticity demand functions are homogeneous of degree zero if and only if the *exponents on income and all prices sum to zero*. This provides a shortcut check for constant-elasticity demand functions.

In applied work, we can guarantee that a demand function satisfies homogeneity if we simply express income and prices in real (or "constant dollar") terms. To do this, simply divide prices and income by some index of the overall price level, such at the Consumer

Price Index published by the U.S. Bureau of Labor Statistics.[3] For example, the following demand function is homogeneous of degree zero, because if p_1, I, *and the CPI* all increase by the same percentage, the quantity demanded of good 1 does not change:

(5.8) $$q_1{}^* = 5 - 3\left(\frac{p_1}{CPI}\right) + 2\left(\frac{I}{CPI}\right) .$$

Purported demand functions that do not satisfy the budget constraint and the homogeneity property cannot be rooted in utility-maximizing behavior. To check whether a function can be a legitimate individual demand function, we can check whether these two properties hold.

Section 5.3: Finding demand functions

Particular choices versus general functions. In the previous chapter, we learned to solve for the consumer's most preferred affordable bundle. Our approach was to solve the budget line and the tangency condition together. Because we assumed particular values for the prices and income, we were able to calculate particular quantities chosen. We now use the same approach but leave prices and income as variables, and so find the consumer's demand *functions*.

Demand functions, like particular choices, depend on the consumer's particular preferences or utility (through the tangency condition). For some utility functions, demand functions are difficult or impossible to find explicitly. However, for a number of realistic utility functions, demand functions can be found without great difficulty.

Example (1). Suppose a consumer has the following Cobb-Douglas utility function $U(q_1,q_2) = q_1^2\, q_2$. What are the demand functions for good #1 and good #2? The tangency condition here is given by $MU_2/MU_1 = q_1/(2q_2) = p_2/p_1$, which after cross-multiplying becomes $q_1p_1 = 2q_2p_2$. Substituting this into the equation for the budget line $I = q_1p_1 + q_2p_2$ and solving yields $q_2^* = I/(3p_2)$. Substituting q_2^* into the budget line (that is, exploiting the adding-up property) and solving yields $q_1^* = 2I/(3p_1)$. (A detailed step-by-step solution is given in an appendix to this chapter.)

Example (2). Suppose a consumer has the following Stone-Geary utility function $U(q_1,q_2) = q_1^2\,(q_2\text{-}3)$. What are the demand functions for good #1 and good #2? The tangency condition here is given by $MU_2/MU_1 = q_1/(2q_2\text{-}6) = p_2/p_1$, which after cross-multiplying becomes $q_1p_1 = 2q_2p_2 - 6p_2$. Substituting (q_1p_1) into the equation for the budget line $I = q_1p_1 + q_2p_2$ and solving yields $q_2^* = I/(3p_2) + 2$. Substituting q_2^* into the budget line (that is, exploiting the adding-up property) and solving yields $q_1^* = 2I/(3p_1) - 2p_2/p_1$. (A detailed step-by-step solution is given in an appendix to this chapter.)

[3] For description and data, see http://www.bls.gov . Similar indexes are available for most countries.

Example (3). Suppose a consumer has the following CES utility function $U(q_1,q_2) = -(1/q_1) - (2/q_2)$. What are the demand functions for good #1 and good #2? The tangency condition here is given by $MU_2/MU_1 = 2q_1^2/q_2^2 = p_2/p_1$, which can be rewritten as $q_2 = q_1 (2p_1/p_2)^{1/2}$. Substituting this into the equation for the budget line $I = q_1p_1 + q_2p_2$ and solving yields, after some tedious algebra, $q_1^* = I/\left(p_1 + \sqrt{2p_1p_2}\right)$ and $q_2^* = I/\left(p_2 + \sqrt{0.5p_1p_2}\right)$. (A detailed step-by-step solution is given in an appendix to this chapter.)

Section 5.4: Effect of income on quantity demanded

A change in income. Recall from chapter 4 that if income rises but prices remain constant, then the budget line shifts up and out, away from the origin, in parallel fashion. Its slope, the price ratio, does not change, but both intercepts rise in proportion to the increase in income. After such a shift, the consumer will choose a bundle on the new budget line (see figure 5.1). Put differently, the increase in income results in an increase in spending. It follows that the consumer will purchase more of one or both goods. Moreover, the consumer will now reach a higher indifference curve, representing a higher level of utility or well-being.

Income expansion path. Imagine the consequences of a continuous increase in income. The consumer will choose a continuous sequence of bundles, each one at a point of tangency between the current budget line and the highest attainable indifference curve. The resulting curve is called the consumer's *income expansion path*,[4] shown as a dotted arrow in figure 5.2. The equation for the income expansion path is simply the property shared by all bundles on that path—namely the tangency condition introduced in the previous chapter.

(5.9) $$\frac{MU_2}{MU_1} = \frac{p_2}{p_1}$$

Example: Suppose a consumer has the utility function $U(q_1,q_2) = q_1^2 q_2^3$ and faces prices $p_1=\$5$ and $p_2 = \$3$. What is the equation for the consumer's income-expansion path? Here, $MU_2 = \partial U/\partial q_2 = 3q_1^2q_2^2$ and $MU_1 = \partial U/\partial q_1 = 2q_1q_2^3$. Applying equation (5.9), yields $(3q_1)/(2q_2) = 3/5$ or $q_1 = (2/5)q_2$.

In this example, the income-expansion path turned out to be a straight line through the origin.[5] Not all income-expansion paths have this simple shape.

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[4] Also called the "income consumption path" or the "income offer curve."

[5] Utility functions are called *homothetic* if they yield expansion paths that are always straight lines through the origin. It can be proven that all utility functions which are homogeneous are also homothetic. For example, the utility functions in examples 4, 6, and 7 are homogeneous of degrees 3, -1, and 5, respectively, so they are all homothetic. However, the utility function in example 5 is not homogeneous, so it is not homothetic—in fact, its expansion paths intersect the q_1 axis at $q_1=3$ instead of passing through the origin.

Figure 5.1. Effect of increase in income

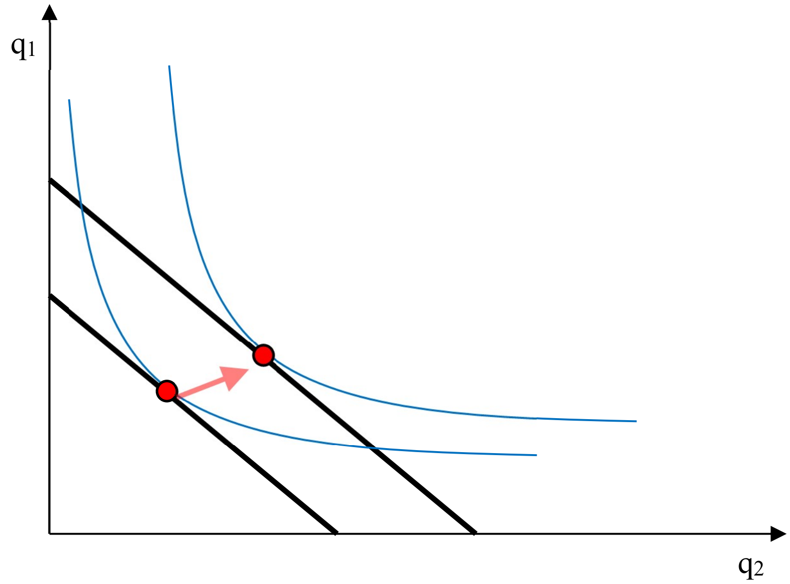
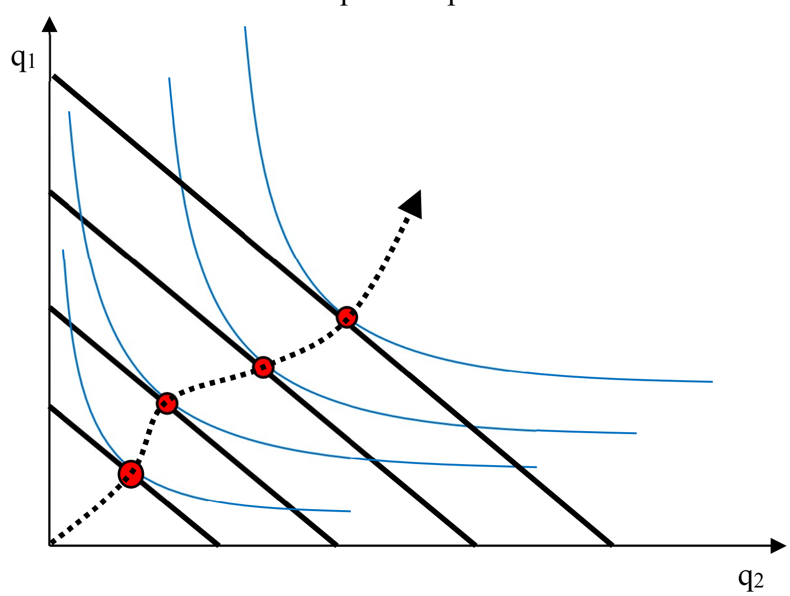


Figure 5.2. Consumer's income expansion path



Example: Suppose a consumer has the utility function $U(q_1,q_2) = q_1 (q_2-3)$ and faces prices $p_1=\$3$ and $p_2 = \$4$. What is the equation for the consumer's income-expansion path? Here, $MU_2 = \partial U/\partial q_2 = q_1$ and $MU_1 = \partial U/\partial q_1 = q_2-3$. Applying equation (5.9), yields $q_1/(q_2-3) = 4/3$ or $q_1 = (4/3)q_2 - 4$. This income-expansion path is *not* a straight line through the origin.

Example: Suppose a consumer has the utility function $U(q_1,q_2) = -(1/q_1) - (1/q_2)$ and faces prices $p_1=\$2$ and $p_2 = \$3$. What is the equation for the consumer's income-expansion path? Here, $MU_2 = \partial U/\partial q_2 = 1/q_2^2$ and $MU_1 = \partial U/\partial q_1 = 1/q_1^2$. Applying equation (5.9), yields $q_1^2/q_2^2 = 3/2$ or $q_1 = \sqrt{3/2}\, q_2$. This income-expansion path is a straight line through the origin.

Normal versus inferior goods. The income-expansion path shown in figure 5.2 slopes upward, indicating that the consumer purchases more of both goods as her or his income rises. However there is no mathematical reason why this need always be the case. Figure 5.3 shows a different consumer who purchases more units of good #2 but fewer units of good #1 as her or his income rises. If a consumer purchases more units of a good as her or his income rises, that good is called a *normal good*. If a consumer purchases fewer units, that good is called an i*nferior good*.

Inferior goods are rare. Broad categories of goods like clothing, transportation, or food are not inferior goods. Inferior goods tend to be narrow, low-quality categories of goods that consumers abandon in favor of high-quality alternatives as their income rises. Examples of inferior goods include used clothing, mass transit, and macaroni-and-cheese dinners. Whether a good is normal or inferior, of course, may vary with individual preferences. Some consumers might have no inferior goods. However, every consumer must have at least one normal good—otherwise spending could not keep pace with rising income.

Engel curves and partial derivatives. A graph of the quantity demanded of a single good against income, holding all prices constant, is called an Engel curve (see figure 5.4). A good is a normal good if its Engel curve slopes up and an inferior good if its Engel curve slopes down. In figure 5.4, good A is a normal good at all levels of income, while good B becomes an inferior good for sufficiently high levels of income.

The slope of an Engel curve is the increase in quantity demanded when income is increased by one dollar, holding all prices constant. This slope is given by the partial derivative of the demand function with respect to income. For a normal good, the partial derivative is positive: $\partial q_1^*/\partial I > 0$. For an inferior good, the partial derivative is negative: $\partial q_1^*/\partial I < 0$.

Example: Suppose the demand for good #1 is given by $q_1^* = I/(2p_1)$. Is good #1 a normal good or an inferior good? First, note that $\partial q_1^*/\partial I = 1/(2p_1)$, which is necessarily *positive*, assuming p_1 is positive. So an increase in income always brings an increase in the quantity demanded. It follows that good #1 is a *normal good*.

Example: Suppose the demand for good #1 is given by $q_1^* = 2I/(3p_1) + 2p_2/p_1$. Is good #1 a normal good or an inferior good? Note that $\partial q_1^*/\partial I = 2/(3p_1)$, which is necessarily *positive*, assuming p_1 is positive. So an increase in income always brings an increase in the quantity demanded. It follows that good #1 is a *normal good*.

Figure 5.3. Income expansion path with one inferior good

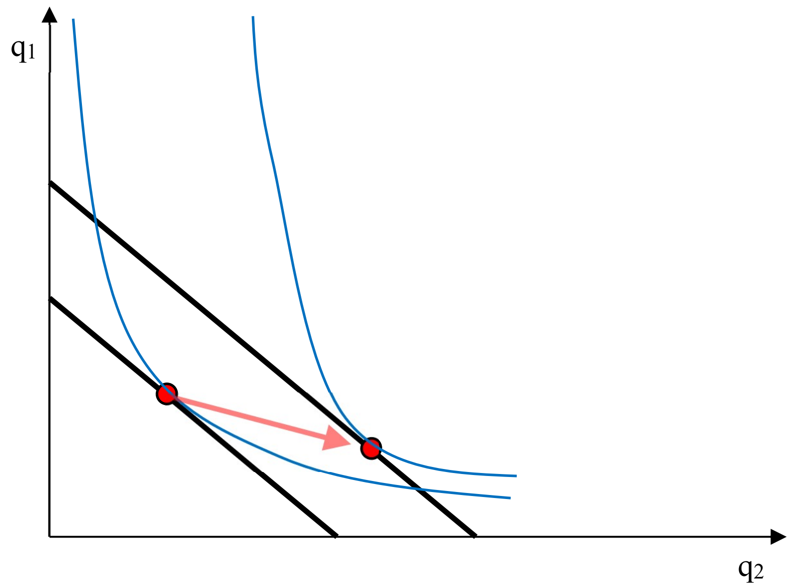
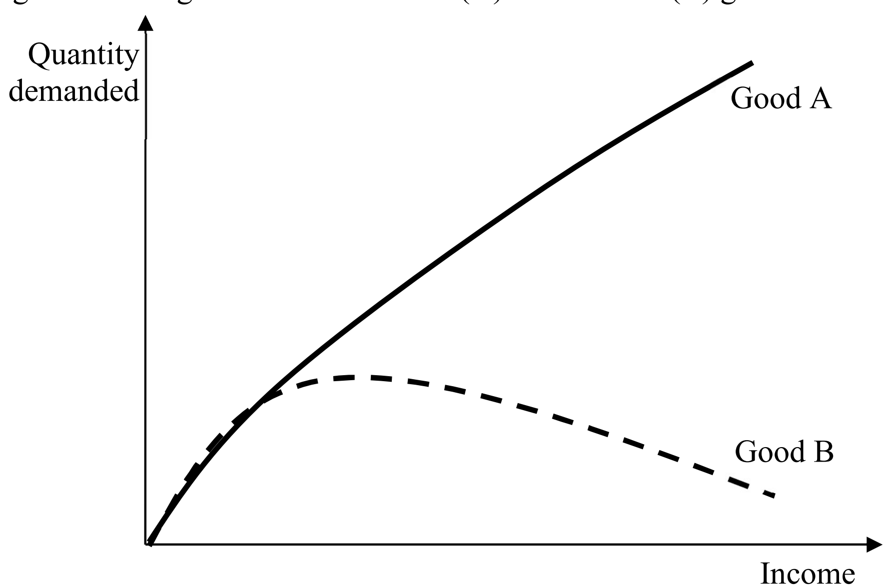


Figure 5.4. Engel curves for normal (A) and inferior (B) goods



Example: Suppose the demand for good #1 is given by $q_1^* = q_1^* = I / (p_1 + (2p_1p_2)^{1/2})$. Is good #1 a normal good or an inferior good? Note that $\partial q_1^*/\partial I = 1/(p_1 + (2p_1p_2)^{1/2})$, which is necessarily *positive*, assuming p_1 is positive. So an increase in income always brings an increase in the quantity demanded. It follows that good #1 is a *normal good*.

Section 5.5: Effect of own price on quantity demanded

A change in the good's own price. If the price of good #1 falls, while income and the other price remain constant, then the budget line rotates out, away from the origin, but anchored on the axis of the good whose price does not change. Assuming good #1 is on the vertical axis, the budget line becomes steeper. After such a shift, the consumer will choose a bundle on the new budget line (see figure 5.5). Since the new budget line is farther from the origin, the consumer will purchase more of one or both goods. Moreover, the consumer will now reach a higher indifference curve, representing a higher level of utility or well-being.

Ordinary versus Giffen goods. Consider the consumer's purchases of good #1, the good whose price has decreased. Figure 5.5 seems to indicate that this consumer has responded to the price decrease by purchasing more of good #1, but there is no mathematical reason why this need always be the case. Figure 5.6 shows a different consumer who purchases fewer units of good #1 as its price decreases. If a consumer purchases more units of good as its price decreases, that good is called an *ordinary good*. If a consumer purchases fewer units, that good is called a *Giffen good*.

Ordinary goods are so common that the negative relationship between the quantity demanded of a good and its own price is often called the "Law of Demand." By contrast, Giffen goods are extremely rare in the world, if any exist at all. We will show later that Giffen goods are in theory a special, extreme case of inferior goods.

Demand curves and partial derivatives. A graph of the quantity demanded of a single good against its own price, holding income and other prices constant, is called a demand curve (see figure 5.7). It is traditional in economics to place price on the vertical axis and the quantity of the same good on the horizontal axis. Regardless of the choice of axes, a good is an ordinary good if its demand curve slopes down and a Giffen good if its demand curve slopes up.

With price on the vertical axis, the slope of the demand curve is the change in price required to increase the quantity demanded by one unit. The reciprocal of the slope is given by the partial derivative of the demand function with respect to the good's own price. For an ordinary good, the partial derivative is negative: $\partial q_1^*/\partial p_1 < 0$. For a Giffen good, the partial derivative would be positive: $\partial q_1^*/\partial p_1 > 0$.

Example: Suppose the demand for good #1 is given by $q_1^* = I/(2p_1)$. Is good #1 an ordinary good or a Giffen good? First, note that $\partial q_1^*/\partial p_1 = -I/(2p_1^2)$, which is necessarily *negative*, assuming I and p_1 are positive. So an increase in price always brings a decrease in the quantity demanded. It follows that good #1 is an *ordinary* good, following the "Law of Demand."

Figure 5.5. Effect of decrease in price of good #1

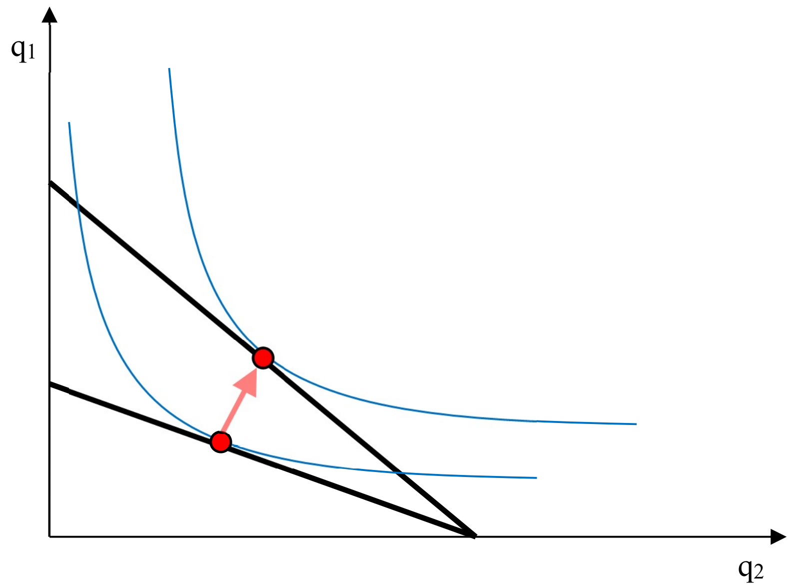


Figure 5.6. Effect of decrease in price of good #1 when good #1 is a Giffen good

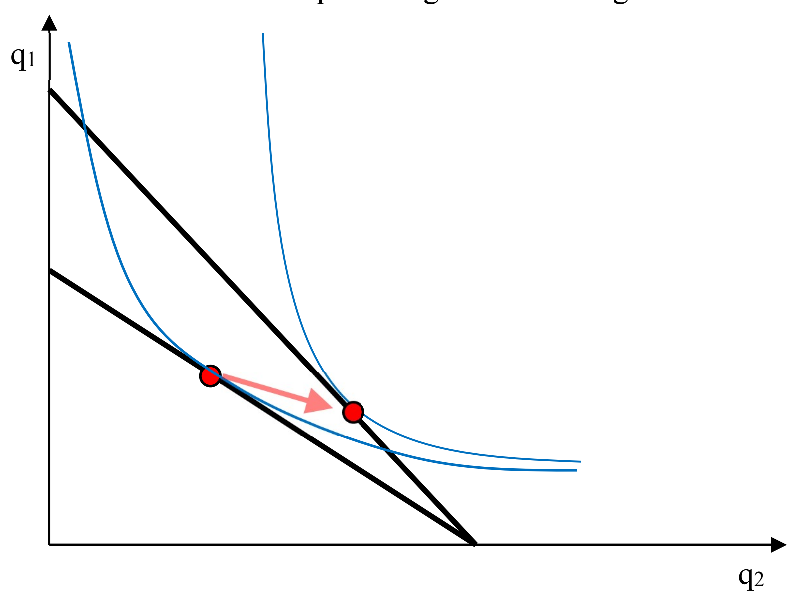
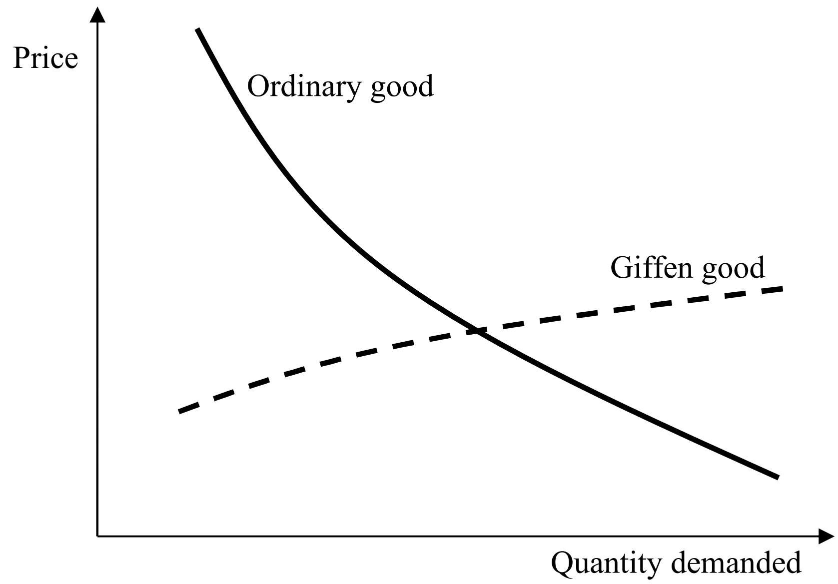


Figure 5.7. Demand curves for ordinary and Giffen goods



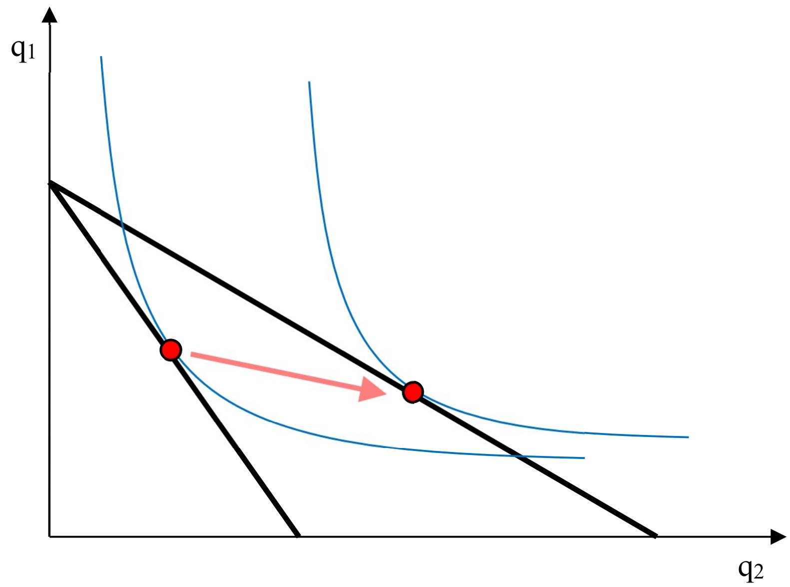
Example: Suppose the demand for good #1 is given by $q_1^* = 2I/(3p_1) + 2p_2/p_1$. Is good #1 a normal good or an inferior good? Note that $\partial q_1^*/\partial p_1 = -2I/(3p_1^2) - 2p_2/p^2$, which is necessarily *negative*, assuming I, p_1, and p_2 are positive. So an increase in p_1 always brings a decrease in the quantity demanded. It follows that good #1 is an *ordinary* good, following the "Law of Demand."

Example: Suppose the demand for good #1 is given by $q_1^* = q_1^* = I / (p_1 + (2p_1p_2)^{1/2})$. Is good #1 a normal good or an inferior good? Note that, using the chain rule, $\partial q_1^*/\partial p_1 = -[I/(p_1 + (2p_1p_2)^{1/2})^2]\ [1 + (1/2)(2p_1p_2)^{-1/2}]\ [2p_2]$. This expression is complicated, but note that each expression in brackets is necessarily positive, assuming I, p_1, and p_2 are positive. Since the whole expression begins with a negative sign, the whole expression is necessarily *negative*. So an increase in p_1 always brings an increase in the quantity demanded. It follows that good #1 is an *ordinary good*, following the "Law of Demand."

Section 5.6: Effect of another price on quantity demanded

A change in another price. If the price of good #2 falls, while income and the other price remain constant, then the budget line rotates out, away from the origin, but anchored on the axis of good #1. Assuming good #2 is on the horizontal axis, the budget line becomes flatter. After such a shift, the consumer will choose a bundle on the new budget line (see figure 5.8). Again, the consumer will purchase more of one or both goods and will now reach a higher indifference curve.

Figure 5.8. Effect of decrease in price of good #2: the case of substitutes

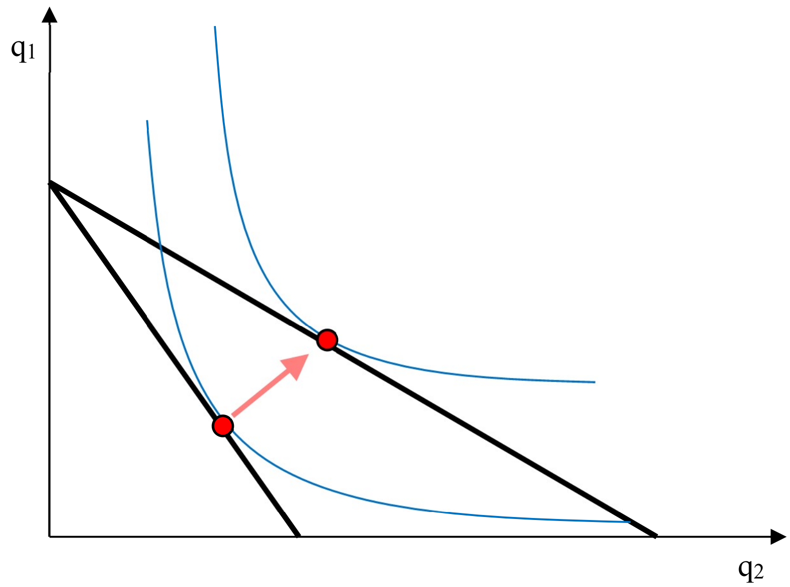


Substitutes versus complements. Consider the consumer's purchases of good #1, the good whose price has not changed. Figure 5.8 seems to indicate that this consumer has responded to the decrease in the price of good #2 by purchasing less of good #1, but there is no mathematical reason why this need always be the case. Figure 5.9 shows a different consumer who purchases more units of good #1 as the price of good #2 decreases. If a consumer purchases fewer units of good as the price of another good decreases, the goods are called *substitutes*. If a consumer purchases more units, the goods are called *complements*.[6]

Substitutes, as the name suggests, are typically goods consumed in place of each other. Similar foods—like oranges and grapefruits—are substitutes. Different forms of energy—like electricity and natural gas—are substitutes. Different brands of the same item—like Chevrolet cars and Ford cars—are close substitutes. By contrast, complements are typically goods consumed together. Computer hardware and software are complements. Large automobiles and gasoline are complements.

[6] More precisely, these are the definitions of *gross substitutes* and *gross complements*. Alternative concepts, called *net substitutes* and *net complements,* are defined with respect to movements along the same indifference curve.

Figure 5.9. Effect of decrease in price of good #2: the case of complements



Partial derivatives. The sign of the derivative of the demand function for one good with respect to the price of another good reveals whether the goods are substitutes or complements. For substitutes, the partial derivative is positive: $\partial q_1^*/\partial p_2 > 0$. For complements, the partial derivative is negative: $\partial q_1^*/\partial p_2 < 0$. Of course, many goods are unrelated in demand. For unrelated goods, the partial derivative is zero: $\partial q_1^*/\partial p_1 = 0$.[7]

Example: Suppose the demand for good #1 is given by $q_1^* = I/(2p_1)$. Are goods #1 and #2 substitutes, complements, or unrelated? First, note that $\partial q_1^*/\partial p_2 = 0$, since p_2 does not appear in the demand function. So p_2 has no effect on q_1. Goods #1 and #2 are *unrelated*.

Example: Suppose the demand for good #1 is given by $q_1^* = 2I/(3p_1) + 2p_2/p_1$. Are goods #1 and #2 substitutes, complements, or unrelated? Note that $\partial q_1^*/\partial p_2 = 2/p_1$, which is necessarily positive, assuming p_1 is *positive*. So an increase in p_2 always brings an increase in the quantity demanded of good #1. It follows that goods #1 and #2 are *substitutes*.

Example: Suppose the demand for good #1 is given by $q_1^* = q_1^* = I / (p_1 + (2p_1p_2)^{1/2})$. Are goods #1 and #2 substitutes, complements, or unrelated? Note that, using the chain rule, $\partial q_1^*/\partial p_2 = -[I/(p_1 + (2p_1p_2)^{1/2})^2]\ [(1/2)(2p_1p_2)^{-1/2}]\ [2p_1]$. This expression is complicated, but note that each expression in brackets is necessarily positive, assuming I, p_1, and p_2 are positive. Since the whole expression begins with a negative sign, the whole expression is necessarily *negative*. So an increase in p_1 always brings an

[7] The definitions of gross substitutes and complements are sometimes ambiguous. For example, sometimes $\partial q_1^*/\partial p_2 < 0$, but $\partial q_2^*/\partial p_1 = 0$, as in example 5 above. It can be proven that this ambiguity does not occur in the definitions of net substitutes and complements.

decrease in the quantity demanded of good #1. It follows that goods #1 and #2 are *complements.*

Section 5.7: Summary

Holding preferences constant, a consumer's choice of quantities to purchase is determined by her or his income and the prices she or he faces. Functions relating these quantities to income and prices are called *demand functions*. Demand functions must satisfy an adding-up property, which insures that spending equals income, and homogeneity of degree zero, which ensures that pure inflation has no effect on quantities demanded. For a number of realistic utility functions, demand functions can be found using the budget line and the tangency condition. The partial derivative of a demand function with respect to income is positive for normal goods (the most common case) and negative for inferior goods. The partial derivative with respect the good's own price is negative for ordinary goods and positive for Giffen goods. However, real-world Giffen goods are rare or nonexistent. The partial derivative with respect to another good's price is positive for substitutes, negative for complements, and zero for unrelated goods.

Appendix to Chapter 5: Demand

Detailed step-by-step solutions for Examples (1), (2), and (3) given in Section 5.3: "Finding demand functions"

Example (1): The consumer is assumed to have utility function $U(q_1,q_2) = q_1^2 q_2$. We seek formulas for the demand functions $q_1^* = q_1^*(p_1,p_2,I)$ and $q_2^* = q_2^*(p_1,p_2,I)$. Our method is to solve the budget line and the tangency condition jointly, treating p_1, p_2, and I as fixed constants. Thus we have two equations (the budget line and the tangency condition) in two unknowns (q_1 and q_2).

The *budget line* is the same as always: $\mathbf{p_1 q_1 + p_2 q_2 = I}$.

The *tangency condition* says that the slope of the budget line must equal the slope of the indifference curve. The slope of the budget line, with q_1 on the vertical axis and q_2 on the horizontal axis, is as usual negative p_2/p_1. The slope of the indifference curve is the negative of the consumer's marginal rate of substitution in consumption (MRSC), whose formula depends on the utility function. The formula for the MRSC is found the usual way: $MRSC = MU_2/MU_1$, where the marginal utilities MU_2 and MU_1 are the partial derivatives of the utility function. For this particular utility function, $MU_2 = \partial U/\partial q_2 = q_1^2$ and $MU_1 = 2q_1 q_2$, so $MRSC = q_1^2 / (2q_1 q_2) = q_1/(2q_2)$. So the tangency condition for this utility function is $\mathbf{q_1/(2q_2) = p_2/p_1}$.

Now we must solve these two equations jointly. There are many ways to do this. One way to begin is to multiply both sides of the tangency condition by $(q_2 p_1)$ to get

$$p_1q_1/2 = p_2 q_2 .$$

Then we can substitute the left-hand side for (p_2q_2) in the equation for the budget line:

$$p_1 q_1 + p_2 q_2 = I$$
$$p_1 q_1 + (p_1q_1/2) = I$$
$$(3/2)\, p_1 q_1 = I$$

Dividing both sides by $(3/2)p_1$ gives the demand function for good #1:

$$\mathbf{q_1^* = (2I)/(3p_1)}$$

Substituting this into the budget line gives

$$p_1 q_1 + p_2q_2 = I$$
$$p_1\, (2I)/(3p_1) + p_2q_2 = I$$
$$(2I/3) + p_2q_2 = I$$

Subtracting $(2I/3)$ from both sides gives

$$p_2q_2 = I/3$$

Dividing both sides by p_2 gives the demand function for good #2:

$$\mathbf{q_2^* = (I)/(3p_2)}$$

Example (2): The consumer is assumed to have utility function $U(q_1,q_2) = q_1^2 (q_2-3)$. Again, we seek formulas for the demand functions $q_1^* = q_1^*(p_1,p_2,I)$ and $q_2^* = q_2^*(p_1,p_2,I)$. Our method is to solve the budget line and the tangency condition jointly, treating p_1, p_2, and I as fixed constants. Thus we have two equations (the budget line and the tangency condition) in two unknowns (q_1 and q_2).

The *budget line* is the same as always: $\mathbf{p_1\, q_1 + p_2\, q_2 = I}$.

The *tangency condition* says that the slope of the budget line must equal the slope of the indifference curve. The slope of the budget line, with q_1 on the vertical axis and q_2 on the horizontal axis, is as usual negative p_2/p_1 . The slope of the indifference curve is the negative of the consumer's marginal rate of substitution in consumption (MRSC), whose formula depends on the utility function. The formula for the MRSC is found the usual way: $MRSC = MU_2/MU_1$, where the marginal utilities MU_2 and MU_1 are the partial derivatives of the utility function. For this particular utility function, $MU_2 = \partial U/\partial q_2 = q_1^2$ and $MU_1 = 2q_1\,(q_2\text{-}3)$, so $MRSC = q_1^2 / (2q_1\,(q_2\text{-}3)) = q_1/(2q_2\text{-}6)$. So the tangency condition for this utility function is $\mathbf{q_1/(2q_2\text{-}6) = p_2/p_1}$.

Now we must solve these two equations jointly. There are many ways to do this. One way to begin is to multiply both sides of the tangency condition by $((2q_2\text{-}6)p_1)$ to get

$$p_1q_1 = p_2\,(2q_2\text{-}6).$$

Then we can substitute the right-hand side for (p_1q_1) in the equation for the budget line:

$$p_1\,q_1 + p_2q_2 = I$$
$$p_2\,(2q_2\text{-}6) + p_2q_2 = I$$
$$(3\,p_2q_2) - 6\,p_2 = I$$

Adding $6p_2$ to both sides gives

$$3\,p_2q_2 = I + 6\,p_2$$

Dividing both sides by $(3p_2)$ gives the demand function for good #2:

$$\mathbf{q_2\,^* = (I + 6\,p_2) / (3p_2),}$$

which can also be written as

$$\mathbf{q_2\,^* = I/(3p_2)\;+\;2\,.}$$

Substituting this into the budget line gives

$$p_1\,q_1 + p_2\,q_2 = I$$
$$p_1\,q_1 + p_2(I/(3p_2) + 2) = I$$
$$p_1\,q_1 + (I/3) + 2p_2 = I$$

Subtracting $((I/3) + 2p_2)$ from both sides gives

$$p_1\,q_1 = I - (I/3) + 2p_2\,,$$
$$p_1\,q_1 = (2I/3) \quad + 2p_2\,,$$

Dividing both sides by p_1 gives the demand function for good #1:

$$\mathbf{q_1\,^* = ((2I/3) + 2p_2) / p_1}\,,$$

which may also be written as

$$\mathbf{q_1\,^* = (2I)/(3p_1) + 2p_2/p_1\,.}$$

Example (3): The consumer is assumed to have utility function $U(q_1,q_2) = -(1/q_1) - (2/q_2)$. This utility function looks peculiar at first, because utility is necessarily negative for all positive values of q_1 and q_2. But the key assumptions do not require that *utility* be positive. They only require that *marginal utilities* be positive ("monotonicity") and that the marginal rate of substitution be diminishing, which turn out to hold for this utility function.

Again, we seek formulas for the demand functions $q_1^* = q_1^*(p_1,p_2,I)$ and $q_2^* = q_2^*(p_1,p_2,I)$. Our method is to solve the budget line and the tangency condition jointly, treating p_1, p_2, and I as fixed constants. Thus we have two equations (the budget line and the tangency condition) in two unknowns (q_1 and q_2).

The *budget line* is the same as always: $\mathbf{p_1\, q_1 + p_2\, q_2 = I}$.

The *tangency condition* says that the slope of the budget line must equal the slope of the indifference curve. The slope of the budget line, with q_1 on the vertical axis and q_2 on the horizontal axis, is as usual negative p_2/p_1 . The slope of the indifference curve is the negative of the consumer's marginal rate of substitution in consumption (MRSC), whose formula depends on the utility function. The formula for the MRSC is found the usual way: $MRSC = MU_2/MU_1$, where the marginal utilities MU_2 and MU_1 are the partial derivatives of the utility function. For this particular utility function, $MU_2 = \partial U/\partial q_2 = 2/q_2^2$ and $MU_1 = \partial U/\partial q_1 = 1/q_1^2$ (which are both positive as required by the assumption of monotonicity). So $MRSC = (2/q_2^2) / (1/q_1^2) = 2q_1^2 / q_2^2$ (which diminishes as q_1 decreases and q_2 increases, as required). So the tangency condition for this utility function is $\mathbf{2q_1^2 / q_2^2 = p_2/p_1}$.

Now we must solve these two equations jointly. There are many ways to do this. One way to begin is to divide both sides of the tangency condition by 2, and take square roots:

$$q_1^2 / q_2^2 = p_2/(2p_1) ,$$
$$q_1 / q_2 = (p_2/(2p_1))^{1/2} .$$

Multiply both sides by q_2 to get

$$q_1 = q_2\, (p_2/(2p_1))^{1/2} .$$

Then we can substitute the right-hand side for (q_1) in the equation for the budget line:

$$p_1\, q_1 + p_2 q_2 = I$$
$$p_1\, (q_2\, (p_2/(2p_1))^{1/2}) + p_2 q_2 = I$$

Now simplify the first term on the left side as follows:

$$\begin{aligned} & p_1\, (q_2\, (p_2/(2p_1))^{1/2}) \\ &= p_1\ q_2\ p_2^{1/2}\ p_1^{-1/2} / 2^{1/2} \\ &= q_2\ p_2^{1/2}\ p_1\ p_1^{-1/2} / 2^{1/2} \\ &= q_2\ p_2^{1/2}\ p_1^{1/2} / 2^{1/2} \\ &= q_2\ (p_1\, p_2 / 2)^{1/2} \\ &= q_2\ (0.5\, p_1\, p_2)^{1/2} . \end{aligned}$$

So the budget line becomes

$$q_2\ (0.5\, p_1\, p_2)^{1/2} + p_2 q_2 = I$$

Now, q_2 is a common factor on the left hand side, so we can write

$$q_2\, [p_2 + (0.5\, p_1\, p_2)^{1/2}] = I .$$

Dividing both sides of the equation by the expression in brackets gives
gives the demand function for good #2:

$$q_2^* = I / [p_2 + (0.5\, p_1\, p_2)^{1/2}] ,$$

which may also be written as

$$q_2^* = \frac{I}{p_2 + \sqrt{0.5 p_1 p_2}} .$$

Now earlier we found that

$$q_1 = q_2\, (p_2/(2p_1))^{1/2} .$$

Substitute the demand function for good #2 (q_2*) into this equation to get

$$q_1^* = \left(\frac{I}{p_2 + \sqrt{0.5 p_1 p_2}} \right) \left(p_2 / (2 p_1) \right)^{1/2}$$

$$= \left(\frac{I}{p_2 + \sqrt{0.5 p_1 p_2}} \right) \sqrt{\frac{p_2}{2 p_1}}$$

$$= \left(\frac{I}{p_2 + \sqrt{0.5 p_1 p_2}} \right) \frac{1}{\sqrt{2 p_1 / p_2}}$$

$$= \frac{I}{\left(p_2 + \sqrt{0.5 p_1 p_2} \right) \sqrt{2 p_1 / p_2}}$$

This may be further simplified as follows:

$$q_1^* = \frac{I}{\left(p_2 \sqrt{2 p_1 / p_2} + \sqrt{0.5 p_1 p_2} \sqrt{2 p_1 / p_2} \right)}$$

$$= \frac{I}{\sqrt{p_2^2} \sqrt{2 p_1 / p_2} + p_1}$$

$$= \frac{I}{\sqrt{2 p_1 p_2} + p_1}$$

which may also be written as

$$q_1^* = \frac{I}{p_1 + \sqrt{2 p_1 p_2}} \quad .$$

[end of appendix]

Problems

(5.1) [Budget constraint] Assume there are only two goods and that the demand for good #1 is given by $q_1^* = 2p_2/p_1$. Substitute this into the equation for the budget line to find the formula for the demand for good #2.

(5.2) [Budget constraint] Let I denote the consumer's income, q_1 the quantity of housing chosen, q_2 denote the quantity of other goods chosen, p_1 the price of housing, and p_2 the price of other goods. Assume a consumer always spends one-fourth of her or his income on housing—regardless of income, the price of housing, or the price of other goods.

a. Write an equation representing this assumption.
b. Solve the equation to find the demand function for housing.

(5.3) [Budget constraint and homogeneity] Consider whether the following functions might be legitimate demand functions for an individual consumer.

$$q_1^* = \frac{3I}{4p_1} \qquad \text{and} \qquad q_2^* = \frac{I}{2p_2}$$

a. Is the budget constraint satisfied by this demand system? (Assume there are only two goods.) Show your work, step by step.
b. Are these functions homogeneous of degree zero in income and prices? Show your work, step by step.

(5.4) [Budget constraint and homogeneity] Consider whether the following functions might be legitimate demand functions for an individual consumer.

$$q_1^* = \frac{I}{2p_1} \qquad \text{and} \qquad q_2^* = \frac{2I}{3p_2}$$

a. Is the budget constraint satisfied by this demand system? (Assume there are only two goods.) Show your work, step by step.
b. Are these functions homogeneous of degree zero in income and prices? Show your work, step by step.

(5.5) [Budget constraint and homogeneity] Consider whether the following functions might be legitimate demand functions for an individual consumer.

$$q_1^* = \frac{2I}{3p_1} + 2 \qquad \text{and} \qquad q_2^* = \frac{I}{3p_2} - \frac{2p_1}{p_2}$$

a. Is the budget constraint satisfied by this demand system? (Assume there are only two goods.) Show your work, step by step.
b. Are these functions homogeneous of degree zero in income and prices? Show your work, step by step.

(5.6) [Homogeneity] Are the following functions homogeneous of degree zero in income and prices? Show your work, step by step.

a. $q_1^* = 5 + 0.05\, I - 2p_1 + 4p_2$.
b. $q_1^* = 5\, I^{0.7}\, p_1^{-0.9}\, p_2^{0.2}$.

(5.7) [Homogeneity] Are the following functions homogeneous of degree zero in income and prices? Show your work, step by step.

a. $q_1^* = (I/p_1) - (p_2/p_1)$.
b. $q_1^* = 3\, I^{1.5}\, p_1^{-0.9}\, p_2^{0.2}$.

(5.8) [Homogeneity] Are the following functions homogeneous of degree zero in income and prices? Show your work, step by step.

a. $q_1^* = 2I / (5p_1)$.
b. $q_2^* = 17\; p_1^{-0.1}\; p_2^{-0.8}\; I^{0.4}$.

(5.9) [Homogeneity] Are the following functions homogeneous of degree zero in income and prices? Show your work, step by step.

a. $q_1^* = 35 + 0.2\, I - 3\, p_1 - 0.1\, p_2$.
b. $q_1^* = 23\; p_1^{-0.3}\; p_2^{0.1}\; I^{0.2}$.

(5.10) [Finding demand functions] Suppose a consumer has utility function $U(q_1,q_2) = q_1^{1/3}q_2^{2/3}$ and has income I. The price of good #1 is p_1 and the price of good #2 is p_2.

a. Give the equation for the consumer's budget line. [Hint: income = spending.]
b. Give a formula for the consumer's marginal rate of substitution in consumption (MRSC) of good #2 for good #1. [Hint: This is the |slope| of the consumer's indifference curve when good #1 is on the vertical axis and good #2 is on the horizontal axis.]
c. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I . [Hint: Begin by setting the MRSC equal to p_2/p_1. Solve this equation for q_2. Substitute the resulting expression in the budget line and solve for q_1.]
d. Find an expression for the consumer's demand for good #2 (q_2^*) as a function of p_1 , p_2, and I .

(5.11) [Finding demand functions] Suppose a consumer has utility function $U(q_1,q_2) = q_1^{1/4}\, q_2^{3/4}$ and has income I. The price of good #1 is p_1 and the price of good #2 is p_2.

a. Give the equation for the consumer's budget line. [Hint: income = spending.]
b. Give a formula for the consumer's marginal rate of substitution in consumption (MRSC) of good #2 for good #1. [Hint: This is the |slope| of the consumer's indifference curve when good #1 is on the vertical axis and good #2 is on the horizontal axis.]
c. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I . [Hint: Begin by setting the MRSC equal to p_2/p_1. Solve this equation for q_2. Substitute the resulting expression in the budget line and solve for q_1.]
d. Find an expression for the consumer's demand for good #2 (q_2^*) as a function of p_1 , p_2, and I .

(5.12) [Finding demand functions] Suppose a consumer has utility function $U(q_1,q_2) = (q_1-15)\, q_2^2$ and has income I. The price of good #1 is p_1 and the price of good #2 is p_2.

a. Give the equation for the consumer's budget line. [Hint: income = spending.]
b. Give a formula for the consumer's marginal rate of substitution in consumption (MRSC) of good #2 for good #1. [Hint: This is the |slope| of the consumer's indifference curve when good #1 is on the vertical axis and good #2 is on the horizontal axis.]
c. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I . [Hint: Begin by setting the MRSC equal to p_2/p_1. Solve this equation for q_2. Substitute the resulting expression in the budget line and solve for q_1.]
d. Find an expression for the consumer's demand for good #2 (q_2^*) as a function of p_1 , p_2, and I .

(5.13) [Finding demand functions] Suppose a consumer has utility function $U(q_1,q_2) = (q_1-5)(q_2-4)$ and has income I. The price of good #1 is p_1 and the price of good #2 is p_2.

a. Give the equation for the consumer's budget line. [Hint: income = spending.]
b. Give a formula for the consumer's marginal rate of substitution in consumption (MRSC) of good #2 for good #1. [Hint: This is the |slope| of the consumer's indifference curve when good #1 is on the vertical axis and good #2 is on the horizontal axis.]
c. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I . [Hint: Begin by setting the MRSC equal to p_2/p_1. Solve this equation for q_2. Substitute the resulting expression in the budget line and solve for q_1.]
d. Find an expression for the consumer's demand for good #2 (q_2^*) as a function of p_1 , p_2, and I .

(5.14) [Finding demand functions] Suppose a consumer has utility function $U(q_1,q_2) = q_1^{1/2} + q_2^{1/2}$ and has income I. The price of good #1 is p_1 and the price of good #2 is p_2.

a. Give the equation for the consumer's budget line. [Hint: income = spending.]
b. Give a formula for the consumer's marginal rate of substitution in consumption (MRSC) of good #2 for good #1. [Hint: This is the |slope| of the consumer's indifference curve when good #1 is on the vertical axis and good #2 is on the horizontal axis.]
c. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I . [Hint: Begin by setting the MRSC equal to p_2/p_1. Solve this equation for q_2. Substitute the resulting expression in the budget line and solve for q_1.]
d. Find an expression for the consumer's demand for good #2 (q_2^*) as a function of p_1 , p_2, and I .

(5.15) [Finding demand functions] The following three utility functions must yield exactly the same demand functions:

$$U(q_1,q_2) = -(1/q_1) - (1/q_2) ,$$
$$V(q_1,q_2) = -(2/q_1) - (2/q_2) ,$$
$$W(q_1,q_2) = -(10/q_1) - (10/q_2) + 5 .$$

Explain why, without solving explicitly for the demand functions.

(5.16) [Linear expenditure system] Suppose a consumer has the utility function $U(q_1,q_2) = (q_1-a)^b (q_2-c)^d$, where a, b, c, and d are arbitrary constants.

a. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I .
b. Find an expression for the consumer's *spending* on good #1 ($p_1q_1^*$) as a function of p_1 , p_2, and I .
c. Explain why this utility function and its associated demand functions are sometimes called a "linear expenditure system."

(5.17) [Cobb-Douglas utility] Suppose a consumer has the utility function $U(q_1,q_2) = q_1^a q_2^b$, where a and b are arbitrary constants.

a. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I .
b. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I .
c. Find an expression for $\ln(q_1^*/q_2^*)$ as a function of p_1 , p_2, and I .
d. The so-called "elasticity of substitution" is defined as $\frac{d \ln(q_1^*/q_2^*)}{d \ln(p_2/p_1)}$. Find an expression for the elasticity of substitution.

(5.18) [CES utility] Suppose a consumer has the utility function $U(q_1,q_2) = a q_1^b + q_2^b$, where a and b are arbitrary constants.

a. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I .
b. Find an expression for the consumer's demand for good #1 (q_1^*) as a function of p_1 , p_2, and I .
c. Find an expression for $\ln(q_1^*/q_2^*)$ as a function of p_1 , p_2, and I .
d. The so-called "elasticity of substitution" is defined as $\frac{d \ln(q_1^*/q_2^*)}{d \ln(p_2/p_1)}$. Find an expression for the elasticity of substitution in terms of a and/or b .
e. Explain why $U(q_1,q_2) = a q_1^b + q_2^b$ is called a "CES" (constant elasticity of substitution") utility function.

(5.19) [Expansion path] Suppose a consumer has the utility function $U(q_1,q_2) = -(1/q_1) - (1/q_2)$ and faces prices p_1=\$2 and p_2 = \$3. What is the equation for the consumer's income-expansion path?

(5.20) [Properties of demand functions] Suppose the purported demand function for good #1 is supposed to be given by $q_1^* = (1/2)\, I\, p_1^{-2/3}\, p_2^{-1/3}$.

a. Is this function homogeneous of degree zero in income and prices? Why or why not?
b. Find an expression for $\partial q_1^*/\partial I$. Is good #1 a normal good or an inferior good? Why?
c. Find an expression for $\partial q_1^*/\partial p_1$. Is good #1 an ordinary good or a Giffen good? Why?
d. Find an expression for $\partial q_1^*/\partial p_2$. Are goods #1 and #2 complements or substitutes? Why?

(5.21) [Properties of demand functions] Suppose the purported demand function for good #1 is supposed to be given by $q_1^* = 5\, I\, p_1^{-4/5}\, p_2^{1/5}$.

a. Is this function homogeneous of degree zero in income and prices? Why or why not?
b. Find an expression for $\partial q_1^*/\partial I$. Is good #1 a normal good or an inferior good? Why?
c. Find an expression for $\partial q_1^*/\partial p_1$. Is good #1 an ordinary good or a Giffen good? Why?
d. Find an expression for $\partial q_1^*/\partial p_2$. Are goods #1 and #2 complements or substitutes? Why?

(5.22) [Properties of demand functions] Suppose the purported demand function for good #1 is supposed to be given by $q_1^* = 3\, (p_1^*)^{-1/2}\, (I^*)$, where $p_1^* = (p_1/CPI)$, $I^* = (I/CPI)$, and CPI = an index of consumer prices. Note that p_2 does not appear in this function, except through the CPI.

a. Is this function homogeneous of degree zero in income and prices? Why or why not?
b. Find an expression for $\partial q_1^*/\partial I$. [Hint: Use the chain rule.] Is good #1 a normal good or an inferior good? Why?
c. Find an expression for $\partial q_1^*/\partial p_1$. [Hint: Use the chain rule.] Is good #1 an ordinary good or a Giffen good? Why?

[end of problem set]