LECTURE NOTES ON MICROECONOMICS

ANALYZING MARKETS WITH BASIC CALCULUS

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Part 1: Mathematical tools

Chapter 2: Introduction to multivariate calculus

> Anyone who understands algebraic notation, reads at a glance in an equation results reached arithmetically only with great labor and pains…
>
> But those skilled in mathematical analysis know that its object is not simply to calculate numbers, but that it is also employed to find the relations between magnitudes which cannot be expressed in numbers and between functions whose law is not capable of algebraic expression…
>
> Antoine Augustin Cournot (1801-1877)

Section 2.1: Functions of several variables

Notation. Often in economics we consider variables that depend on several other variables. For example, the quantity demanded of a good typically depends on the price of the good, the income of consumers, and perhaps the prices of other goods which are substitutes or complements for this good. As a second example, the amount of output in a production process may depend on the quantities of several different inputs, like workers, machines, electricity, raw materials, etc. As a third example, the amount of utility or well-being a person enjoys may depend on the amount consumed of food, clothing, shelter, health care, etc.

If the variable y depends on n different variables $x_1, ..., x_n$ then we can write this function as $y = f(x_1, ..., x_n)$. Here, the x variables, called the *arguments* of the function, are distinguished by subscripts. Subscripts are easier to use than different letters of the alphabet if the function has many arguments, and one could argue that most functions in economics have many arguments. Nevertheless, most of the important insights concerning functions of several variables can be seen from examples with just two arguments.

Examples. Table 2.1 illustrates a function of two variables $y = f(x_1,x_2)$. This is a production function. The number of parts produced per hour y depends on the number of machines x_1 and the number of workers x_2 employed.

Table 2.2 illustrates a different function of two variables. This is a utility function for a consumer. The level of utility or well-being depends on the amount of food consumed and the amount of other goods consumed.

Table 2.1. A production function with two inputs

		Number of workers 1	2	3
Number of machines	*1*	**10 parts per hour**	**16 parts per hour**	**21 parts per hour**
	2	**14 parts per hour**	**18 parts per hour**	**25 parts per hour**
	3	**18 parts per hour**	**22 parts per hour**	**27 parts per hour**

Table 2.2. A utility function

		Units of other goods 10	20	30
Units of food	*10*	**80**	**100**	**120**
	20	**100**	**150**	**170**
	30	**120**	**170**	**200**

Functions of several variables occur everywhere in economics. For example, the quantity demanded of ice cream is a function of the price of ice cream, the price of frozen yogurt, and the income of consumers. The quantity supplied of corn is a function of the price of corn, the price of corn seed, and the price of fertilizer. Investment spending in the macroeconomy is a function of the level of GDP and the interest rate.

Section 2.2: Graphing functions of two variables

The problem of representing three dimensions. How can we represent functions of two arguments graphically? Figure 2.1 shows the production function described by table 2.1 as a three-dimensional graph. Levels of output are graphed here as discrete vertical bars, on the assumption that workers and machines cannot be employed for fractional hours. Figure 2.2 shows another representation of a different production function. Levels of output are graphed as a smooth surface, on the assumption that workers and machines can be employed for fractional hours. These three-dimensional graphs show a wealth of information. But they are difficult to draw without a computer and difficult to read precisely. (For example, how much output is produced by three workers and two machines? The answer is difficult to obtain from these graphs.)

An alternative way of graphing a function of two arguments is to hold one of the three variables constant. This technique reduces a three-dimensional problem to two dimensions, suitable for sketching without the aid of a computer. The next two figures display the same information as figure 2.2, while holding one variable constant.

Figure 2.3 shows the relationship between workers and output, holding the number of machines fixed. This figure shows a cross-sectional view of figure 2.2. Each curve corresponds to a different level of machines.

Alternatively, figure 2.4 shows the relationship between workers and machines as curves, holding the amount of output constant. This figure shows a bird's view of figure 2.2. Each curve corresponds to a different level of output.

Figure 2.1. 3D graph of a production function with two inputs

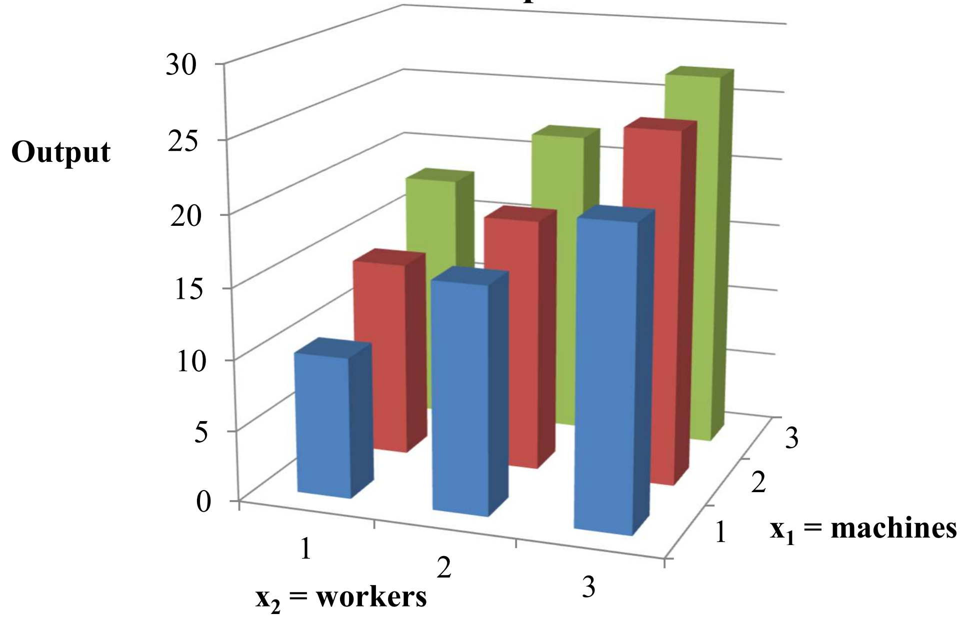


Figure 2.2. Another 3D graph of a production function

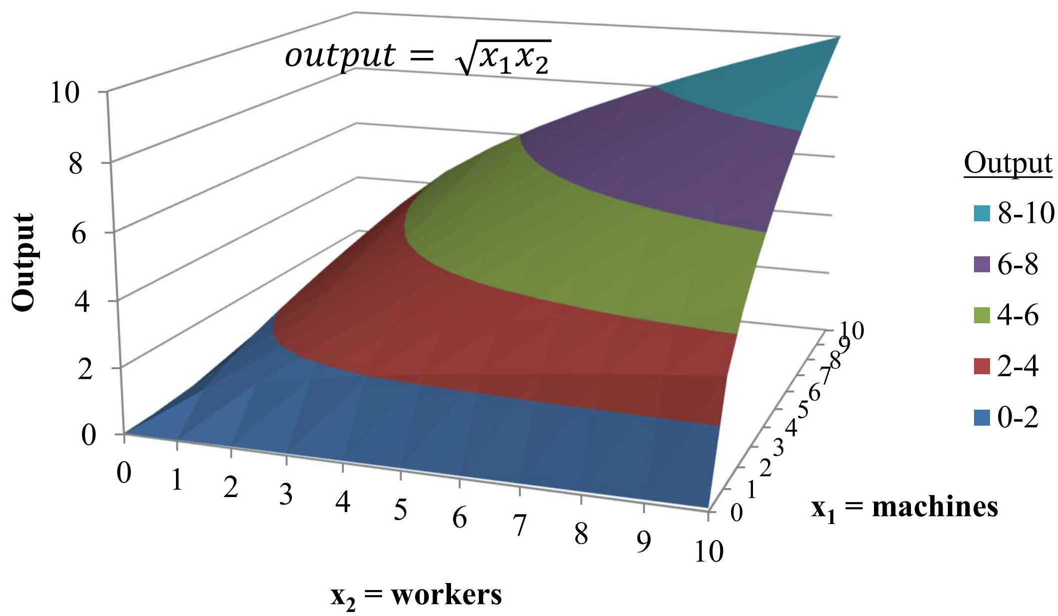


Figure 2.3. Curves of output against workers, holding machines constant

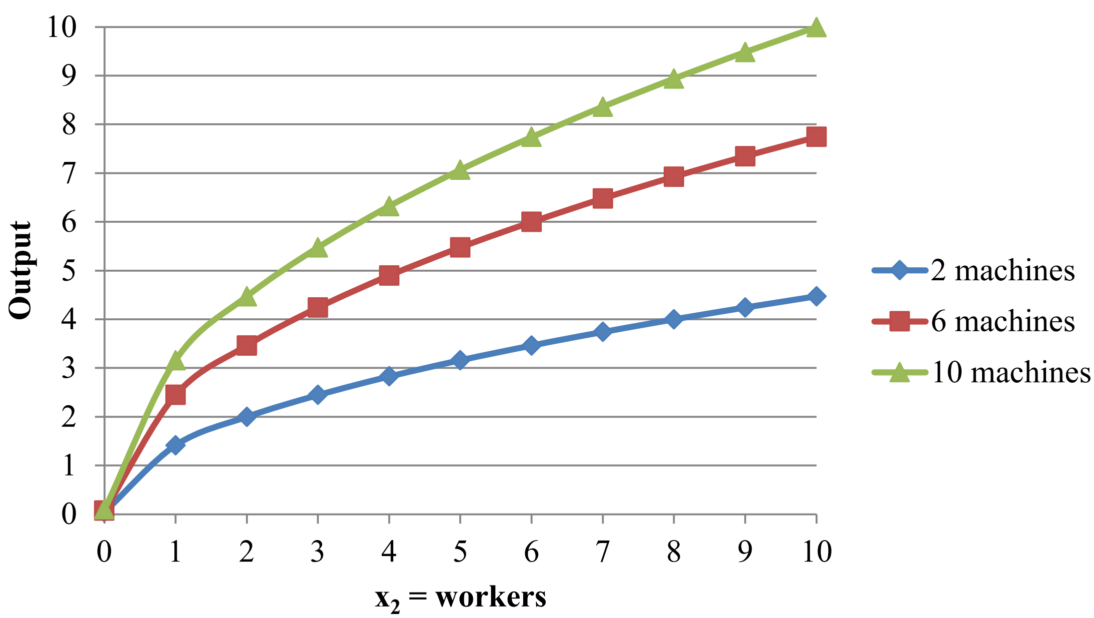
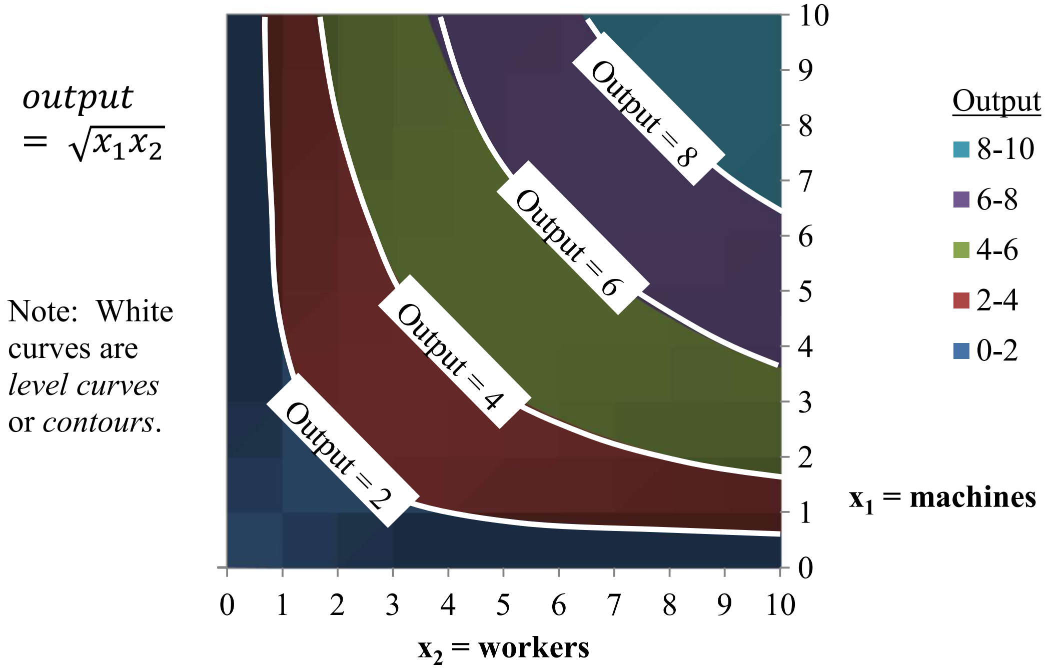


Figure 2.4. Level curves of a production function



Level curves. The curves in figure 2.4 looks quite different from a graph of a function of one variable. Curves of this type—that is, graphs of x_1 against x_2 while holding y constant—are called *level curves*. Graphs of level curves are widely used in microeconomics, but also in other unrelated fields. For example, topographic maps show level curves that connect locations having the same altitude. Cartographers call their level curves *contours*. In our terminology, x_1 is latitude, x_2 is longitude, and y is altitude. Weather maps often show level curves that connect locations having the same temperature. Meteorologists call their level curves *isotherms*. In our terminology, x_1 is latitude, x_2 is longitude, and y is temperature.

Economists have developed a number of forms for functions of several variables that are designed to capture real economic phenomena accurately. Often these forms are referred to by the names of the economists who first used them. Table 2.3 lists some of these forms.

Section 2.3: What is a partial derivative?

Definition of partial derivative. Suppose we have a function $y = f(x_1,x_2)$ describing the relationship between variables x_1, x_2, and y. For example, $f(x_1,x_2)$ might be a production function, with x_1 and x_2 denoting the number of machines and workers employed, or a utility function, with x_1 and x_2 denoting the amounts of food and other goods consumed. To measure the effect of arguments on the function, we must recognize that there is more than one argument. For example, if we want to measure the increase in output that results from adding one more worker to the production process, are we assuming that the number of machines stays constant? Grows proportionately with the number of workers? Decreases as workers are substituted for machines? Obviously the increase in output depends on which assumption we make.

If the number of machines stays constant, we are measuring the slope of the function relating output y to workers x_2, while treating the number of machines x_1 as a fixed constant. That is, we are measuring the slope of a curve like one of those in figure 2.3. The slope of the tangent line is a derivative, but not of the simple kind defined in the previous chapter. This derivative measures only partially the effects of inputs on output, since only one input is permitted to vary. Hence it is termed a partial derivative, and the special symbol "∂" is used, rather than a simple "d." The formal definition of the *partial derivative of* y *with respect to* x_2 is this:

$$(2.1) \qquad \lim_{\Delta x_2 \to 0} \frac{f(x_1, x_2 + \Delta x_2)}{\Delta x_2} = \frac{\partial f}{\partial x_2} = \frac{\partial y}{\partial x_2}.$$

Clearly, the slopes of the curves in figure 2.3 vary. At the same value of x_2, the derivative $\partial f/\partial x_2$ will vary, depending on the assumed value of x_1, the number of machines.

Table 2.3. Examples of functions of two variables, $f(x_1, x_2)$

Name	General form	Examples
Linear function	$y = a x_1 + b x_2$	$y = 5 x_1 + 3 x_2$ $y = 20 x_1 + 10 x_2$
Quasi-linear function	$y = a x_1 + f(x_2)$	$y = 3 x_1 + \ln(x_2)$ $y = 2 x_1 + x_2^{1/2}$
Generalized linear function[i]	$y = a x_1 + b x_2 + c (x_1 x_2)^{1/2}$	$y = 2 x_1 + 6 x_2 + 3 (x_1 x_2)^{1/2}$ $y = 5 x_1 + 4 x_2 + (x_1 x_2)^{1/2}$
Fixed-proportions	$y = \min\{ a x_1, b x_2\}$	$y = \min\{ x_1, 5 x_2\}$ $y = \min\{ 3 x_1, 2 x_2\}$
Cobb-Douglas function[ii]	$y = a x_1^b x_2^c$	$y = 2 x_1^2 x_2^3$ $y = 5 x_1^{1/3} x_2^{2/3}$
Stone-Geary function[iii]	$y = a (x_1 - b)^c (x_2 - d)^e$	$y = (x_1 - 5)^{1/4} (x_2 - 7)^{3/4}$ $y = 4 x_1^2 (x_2 - 3)$
Addilog[iv]	$y = a x_1^b + c x_2^d$	$y = 2 x_1 + x_2^{1/2}$ $y = 3 x_1 + 4 x_2^{2/3}$

CES function[v]	$y = (a\, x_1^b + c\, x_2^b)^d$	$y = (2\, x_1^{1/2} + 3\, x_2^{1/2})^2$ $y = 3\, x_1^{1/2} + 5\, x_2^{1/2}$ $y = -2\, x_1^{-1} - 3\, x_2^{-1}$

[i] R.G.D. Allen, *Mathematical analysis for economists*, New York: St. Martin's Press, 1960, p. 325, question 34. See also W.E. Diewert, "An application of the Shephard duality theorem: a generalized Leontief production function," *Journal of Political Economy, 79*, no. 3 (May-June 1971), pp. 481-507.

[ii] P.Y. Douglas, *The theory of wages*, New York: Macmillan Co., 1934.

[iii] R. C. Geary, "A note on 'A constant utility index of the cost of living,'" *Review of Economic Studies, 18* (1950-51), pp. 65-66.

[iv] H. Houthakker, "Additive preferences," *Eonometrica, 28,* no. 2 (April 1960), pp. 244-257.

[v] K. J. Arrow, H. B. Chenery, B. S. Minhas, and R. M. Solow, "Capital-labor substitution and economic efficiency," *Review of Economics and Statistics*, 43, no. 3, (August 1961), pp. 225-250. Note that in the second and third examples, d equals one.

We can analogously measure the increase in output that results from adding one more machine to the production process, holding the number of workers constant. The formal definition of the partial derivative of y with respect to x_1 is this:

$$\text{(2.2)} \qquad \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2)}{\Delta x_1} = \frac{\partial f}{\partial x_1} = \frac{\partial y}{\partial x_1}.$$

Meaning of the partial derivative. The partial derivative measures the instantaneous rate of change in the function. If x_2 increases by one unit and x_1 is held constant, then y will increase by approximately the value of $\partial f/\partial x_2$, the partial derivative of y with respect to x_2. More generally, if x_2 increases by a small amount Δx_2, and x_1 is held constant, then y will increase by approximately Δx_2 times the value of the partial derivative.

Example: Suppose again that x_1 represents machines, x_2 represents workers, and y represents output. Suppose at some particular values of x_1 and x_2, the partial derivative $\partial f/\partial x_2=0.05$ tons per worker. Then if the number of workers increases by one, output will increase by approximately $\partial f/\partial x_2$ or 0.05 tons. Similarly, if the number of workers increases by two, output will increase by approximately two times $\partial f/\partial x_2$ or 0.1 tons.

The total derivative. This method can also be used to approximate the total effect of small changes in both arguments. Continuing the example just given, suppose also that $\partial f/\partial x_1=0.7$ tons per machine. Then if the number of workers increases by two *and* the number of machines increases by one, output will increase by approximately two times $\partial f/\partial x_2$ plus one times $\partial f/\partial x_1$ or 0.8 tons. In general, the total effect of changes in all arguments can be approximated by summing their individual effects as calculated using partial derivatives:

$$\text{(2.3)} \qquad \Delta y \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 .$$

This approximation is more accurate if the changes are small.

Example: For a particular function $y = f(x_1,x_2)$, suppose at a particular point, the partial derivatives have the values $\partial f/\partial x_1=1/3$ and $\partial f/\partial x_2 = -2$. If x_1 increases by 1, then y will increase by $1/3 \times 1 = 1/3$. If x_2 increases by 1/2, then y will increase by $(-2) \times (1/2) = -1$; in other words, y will decrease by one unit. If both changes happen simultaneously, y will increase by $(1/3)-1 = -2/3$. In other words, y will decrease by (2/3) unit.

Example: For a particular function $y = f(x_1,x_2)$, suppose at a particular point, the partial derivatives have the values $\partial f/\partial x_1=2$ and $\partial f/\partial x_2=3$. Suppose x_2 increases by 1 unit. Suppose we wish to cancel the effect of this increase in x_2 by changing x_1. Should we increase or decrease x_1? By approximately how much? The answer is found with equation (2.3). We are given that $\Delta x_2 = 1$. Since we do not want y to change, $\Delta y = 0$. Substituting these values and the partial derivatives into equation (2.3) yields $0 \approx (2)\Delta x_1 + (3)(1)$. This is easily solved to get $\Delta x_1 = -1.5$; in other words, x_1 must *decrease* by 1.5 units.

Section 2.4: Finding partial derivatives

To find formulas for partial derivatives, we can use the same rules as for ordinary derivatives, taking care to treat other arguments as fixed though unknown constants. Thus, when finding $\partial f/\partial x_1$ from $y = f(x_1,x_2)$, we must treat x_2 as a fixed constant, like 3 or (−1/2). Similarly, when finding $\partial f/\partial x_2$, we must treat x_1 as a fixed constant.

Recall from basic calculus that *additive* constants disappear when we take derivatives. For example, suppose $y = f(x) = x^2 + 7$. Then $dy/dx = 2x + 7$. The sample principle applies when taking partial derivatives. For example, suppose $y = f(x_1,x_2) = x_1^2 + x_2$. When taking the partial derivative with respect to x_1, we must treat x_2 as an additive constant. So $\partial f/\partial x_1 = 2x_1$.

By constrast, *multiplicative* constants must be retained when we take derivatives. For example, suppose $y = f(x) = x^2\ 5$. Then $dy/dx = 2x\ 5 = 10x$. The same principle applies when taking partial derivatives. For example, suppose $y = f(x_1,x_2) = x_1^2x_2$. When taking the partial derivative with respect to x_1, we must treat x_2 as a multiplicative constant. So $\partial f/\partial x_1 = 2x_1x_2$.

Partial derivatives example 1: Suppose $y = f(x_1,x_2) = 2\ x_1 + 7\ x_2$ (a linear function). Then using the rule for derivatives of sums of functions, $\partial f/\partial x_1 = 2$. Similarly, $\partial f/\partial x_2$ =7. (Note that all linear functions must have constant partial derivatives.)

Partial derivatives example 2: Suppose $y = f(x_1,x_2)= (x_1\ x_2)^{1/2}$. Then using the "chain rule" for functions of functions, $\partial f/\partial x_1 = (1/2)\ (x_1\ x_2)^{-1/2}\ x_2 = (1/2)\ (x_2/x_1)^{1/2}$. Similarly, $\partial f/\partial x_2 = (1/2)\ (x_1/x_2)^{1/2}$.

Partial derivatives example 3: Suppose $y = f(x_1,x_2) = 5\ x_1 + 3\ x_2 + 2\ (x_1\ x_2)^{1/2}$ (a generalized linear function). Then using the rule for sums and the chain rule, $\partial f/\partial x_1 = 5 + (x_2/x_1)^{1/2}$. Similarly, $\partial f/\partial x_2 = 3 + (x_1/x_2)^{1/2}$.

Partial derivatives example 4: Suppose $y = f(x_1,x_2) = 5\ x_1^3\ x_2^2$ (a Cobb-Douglas function). Then $\partial f/\partial x_1 = 15\ x_1^2\ x_2^2$ and $\partial f/\partial x_2 = 10\ x_1^3\ x_2$.

Partial derivatives example 5: Suppose $y = f(x_1,x_2) = 6\ x_1^{2/3}\ x_2^{1/3}$ (another Cobb-Douglas function). Then $\partial f/\partial x_1 = 4\ x_1^{-1/3}\ x_2^{1/3}$ and $\partial f/\partial x_2 = 2\ x_1^{2/3}\ x_2^{-2/3}$.

Partial derivatives example 6: Suppose $y = f(x_1,x_2) = 8\ x_1^{1/4}\ (x_2-3)^{3/4}$ (a Stone-Geary function). Then $\partial f/\partial x_1 = 2\ x_1^{-3/4}\ (x_2-3)^{3/4}$. Similarly $\partial f/\partial x_2 = 6\ x_1^{1/4}\ (x_2-3)^{-1/4}$.

Partial derivatives example 7: Suppose $y = f(x_1,x_2) = 6\ x_1^{1/2} + 4\ x_2^{1/2}$ (a CES function). Then $\partial f/\partial x_1 = 3\ x_1^{-1/2}$. Similarly $\partial f/\partial x_2 = 2\ x_2^{-1/2}$.

Partial derivatives example 8: Suppose $y = f(x_1,x_2) = -5\ x_1^{-1} - 3\ x_2^{-1}$ (another CES function). Then $\partial f/\partial x_1 = 5\ x_1^{-2}$. Similarly $\partial f/\partial x_2 = 3\ x_2^{-2}$.

Partial derivatives example 9: Suppose $y = f(x_1,x_2) = (x_1^{-2} + 3\ x_2^{-2})^{-1/2}$ (yet another CES function). Then $\partial f/\partial x_1 = (x_1^{-2} + 3\ x_2^{-2})^{-3/2}\ x_1^{-3}$. Similarly $\partial f/\partial x_2 = (x_1^{-2} + 3\ x_2^{-2})^{-3/2}\ 3x_2^{-3}$.

Section 2.5: Approximation rules for products and ratios

Useful rules of thumb can be derived from equation (2.3).

Products. Suppose $y = x_1 x_2$. Then $\partial f/\partial x_1 = x_2$ and $\partial f/\partial x_2 = x_1$. Applying formula (2.3), $\Delta y \approx x_2 \, \Delta x_1 + x_1 \, \Delta x_2$. Dividing both sides by $(y = x_1 x_2)$ yields

$$(2.4) \qquad \frac{\Delta y}{y} \approx \frac{\Delta x_1}{x_1} + \frac{\Delta x_2}{x_2} .$$

In words, if y is the product of x_1 and x_2, then *the percent change in y is approximately the percent change in x_1 plus the percent change in x_2.*

Example: By definition, output equals labor productivity (output per worker) times the number of workers. Therefore, if labor productivity increases by 2 percent and the number of workers increases by 1 percent, then output increases by about 3 percent. Similarly, revenue is defined as price times quantity. Therefore, if price increases by 5 percent and quantity decreases by 3 percent, revenue increases by about 5−3 = 2 percent.

Ratios. Similarly, suppose $y = (x_1 / x_2)$. Then $\partial f/\partial x_1 = 1/x_2$ and $\partial f/\partial x_2 = -x_1/x_2^2$. Applying formula (2.3), $\Delta y \approx (1/x_2)\Delta x_1 + (-x_1/x_2^2)\Delta x_2$. Dividing both sides by $(y = x_1 / x_2)$ yields

$$(2.5) \qquad \frac{\Delta y}{y} \approx \frac{\Delta x_1}{x_1} - \frac{\Delta x_2}{x_2} .$$

In words, if y is the ratio of x_1 to x_2, then *the percent change in y is approximately the percent change in x_1 minus the percent change in x_2.*

Example: Labor productivity[1] equals output divided by the number of workers. Therefore, if output increases by 4 percent and the number of workers increases by 1 percent, then labor productivity increases by about 3 percent. Alternatively, if output does not change but the number of workers decreases by 2 percent, then productivity increases by about 2 percent.

These approximation rules are accurate for small percent changes

Section 2.6: What is a partial elasticity?

Definition of partial elasticity. The partial effect of one argument on a function can also be measured as a ratio of percent changes. Suppose $y = f(x_1,x_2)$. The formal definitions of the partial elasticities of y with respect to x_1 and with respect to x_2 are as follows.

[1] Formally, the *average product of labor*.

$$(2.5) \qquad \varepsilon_1 = \left(\frac{\partial y}{\partial x_1}\right)\left(\frac{x_1}{y}\right) \qquad \varepsilon_2 = \left(\frac{\partial y}{\partial x_2}\right)\left(\frac{x_2}{y}\right)$$

Meaning of the partial elasticity. The partial elasticity measures the instantaneous rate of change in the function in percentage terms. If x_1 increases by one percent, then y will increase in percent by approximately the value of the partial elasticity of y with respect to x_1 . More generally, if x_1 increases by a small percent $(\Delta x_1/x_1)$, then y will increase by a percent approximately equal to $(\Delta x_1/x_1)$ times the value of the partial elasticity.

Example: Suppose again that x_1 represents machines, x_2 represents workers, and y represents output, and the partial elasticities are $\varepsilon_1 = 0.25$ and $\varepsilon_2 = 0.75$. If the number of machines increases by 4%, then output will increase by approximately 1%. Similarly, if the number of workers increases by 4%, then output will increase by approximately 3%.

No units of measure. Like ordinary elasticities, partial elasticities are pure numbers. They have no units of measure because they represent ratios of percent changes. Partial elasticities can also be defined in terms of the logarithms of the variables, as follows.

$$(2.6) \qquad \varepsilon_1 = \left(\frac{\partial \ln y}{\partial \ln x_1}\right) \qquad \varepsilon_2 = \left(\frac{\partial \ln y}{\partial \ln x_2}\right)$$

The total derivative in elasticity form. The total effect of percent changes in all arguments can be approximated by summing their individual effects. To formalize this, divide both sides of equation (2.3) by y . Then multiply the first term on the right side by (x_1/x_1) and the second term by (x_2/x_2). The result is as follows.

$$(2.7) \qquad \frac{\Delta y}{y} \approx \left(\frac{\partial f}{\partial x_1}\right)\left(\frac{x_1}{y}\right)\frac{\Delta x_1}{x_1} + \left(\frac{\partial f}{\partial x_2}\right)\left(\frac{x_2}{y}\right)\frac{\Delta x_2}{x_2} = \varepsilon_1 \frac{\Delta x_1}{x_1} + \varepsilon_2 \frac{\Delta x_2}{x_2}$$

This approximation is accurate if the percent changes are small.

Example: For a particular function $y = f(x_1,x_2)$, suppose at a particular point, the partial elasticities have the values $\varepsilon_1=2$ and $\varepsilon_2=-1/2$. If x_1 increases by 3 percent, then y increases by approximately $2\times3 = 6$ percent. If x_2 increases by 4 percent , then y increases by approximately $(-1/2)\times4 = -2$ percent; in other words y *decreases* by 2 percent. If both changes happen simultaneously, y increases by $6-2 = 4$ percent.

Section 2.7: Finding partial elasticities of functions

General approach. Using the formal definition of the partial elasticities given in equation (2.5), formulas for partial elasticities can easily be found.

For example, suppose $y = f(x_1,x_2) = 2 x_1 + 7 x_2$. Then $\varepsilon_1 = (2) (x_1) / (2 x_1 + 7 x_2) = (2x_1)/(2 x_1 + 7 x_2)$. Also $\varepsilon_2 = (7x_2) / (2 x_1 + 7 x_2)$.

As another example, suppose $y = f(x_1,x_2) = 5\, x_1^3\, x_2^2$. Then $\varepsilon_1 =$ $(15\, x_1^2\, x_2^2)\,(x_1) / (5\, x_1^3\, x_2^2) = 3$. Also $\varepsilon_1 = (10\, x_1^3\, x_2)\,(x_2) / (5\, x_1^3\, x_2^2) = 2$.

Cobb-Douglas function. The last example is a case where the partial elasticities are constant numbers, not functions of x_1 or x_2. All Cobb-Douglas functions have this property. For suppose $y = a\, x_1^b\, x_2^c$ for fixed numbers a, b, and c. Then $\varepsilon_1 = (b\, a\, x_1^{b-1}\, x_2^c)\,(x_1) / (a\, x_1^b\, x_2^c) = b$. Also $\varepsilon_2 = (c\, a\, x_1^b\, x_2^{c-1})\,(x_2) / (a\, x_1^b\, x_2^c) = c$. For this reason, Cobb-Douglas functions are sometimes called "constant elasticity" functions.

Section 2.8: What is the marginal rate of substitution?

Definition of the marginal rate of substitution. Recall that graphs of x_1 against x_2 while holding y constant—such as those in figure 2.4—are called level curves. The slope of these curves can be calculated using the formula for the total derivative given earlier in equation (2.3). Since y is constant along a level curve, $\Delta y = 0$, so we have the following.

(2.8) $$0 \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 .$$

Now for infinitesimal changes dx_1 and dx_2, this equation holds exactly. Substituting dx_1 and dx_2 for Δx_1 and Δx_2 and solving for their ratio gives the following expression for the slope of a level curve.

(2.9) $$\frac{dx_1}{dx_2} = -\frac{\partial f / \partial x_2}{\partial f / \partial x_1}$$

Note how, on the right side of this equation, the partial derivative with respect to x_1 appears in the *denominator* while the partial derivative with respect to x_2 appears in the *numerator*.

The marginal rate of substitution (MRS) of x_2 for x_1 is defined as the absolute value of the slope of the level curve drawn with x_1 on the vertical axis and x_2 on the horizontal axis. Its formal definition is therefore as follows (assuming both partial derivatives are positive).

(2.10) $$\text{MRS of } x_2 \text{ for } x_1 = \frac{\partial f / \partial x_2}{\partial f / \partial x_1}$$

Meaning of the MRS. The MRS measures the instantaneous rate at which one argument can be substituted for another, without changing the value of y. In particular, if x_2 increases by one unit, then x_1 may be decreased by an amount equal to the MRS of x_2 for x_1, without changing y. More generally, if x_2 increases by a small amount Δx_2, then to keep y constant, x_1 must be decreased by approximately Δx_2 times the value of the MRS. Conversely, if x_2 decreases by a small amount Δx_2, then to keep y constant, x_1 must be increased by approximately Δx_2 times the value of the MRS.

For example, suppose again that x_1 represents machines, x_2 represents workers, and y represents output. Suppose the MRS equals two. Then if an additional worker is added, the number of machines may be reduced by approximately two without changing the amount of output. Conversely, approximately two machines can substitute for one worker without changing the amount of output.

Section 2.9: Finding the marginal rate of substitution

To find formulas for the marginal rate of substitution, we can simply find partial derivatives and form their ratio. Here, the marginal rates of substitution for the examples introduced in section 2.4 are found.

MRS example 1: Suppose $y = f(x_1,x_2) = 2\,x_1 + 7\,x_2$. Then the MRS = 7/2.

MRS example 2: Suppose $y = f(x_1,x_2) = (x_1\,x_2)^{1/2}$. Then the MRS = $\dfrac{(1/2)\left(x_1/x_2\right)^{1/2}}{(1/2)\left(x_2/x_1\right)^{1/2}}$.

But this awkward expression can be simplified further. First, of course, we should cancel the (1/2) factors in the numerator and denominator. Then we can rewrite the MRS as $\dfrac{\left(x_1^{1/2}/x_2^{1/2}\right)}{\left(x_2^{1/2}/x_1^{1/2}\right)} = \left(\dfrac{x_1^{1/2}}{x_2^{1/2}}\right) \times \dfrac{1}{\left(x_2^{1/2}/x_1^{1/2}\right)} = \left(\dfrac{x_1^{1/2}}{x_2^{1/2}}\right) \times \left(\dfrac{x_1^{1/2}}{x_2^{1/2}}\right) = \dfrac{x_1^{1/2} \times x_1^{1/2}}{x_2^{1/2} \times x_2^{1/2}}$. Finally, if two factors have the same base, we multiply them by adding the powers. So $\dfrac{x_1^{1/2} \times x_1^{1/2}}{x_2^{1/2} \times x_2^{1/2}} = \dfrac{x_1}{x_2}$.

MRS example 3: Suppose $y = f(x_1,x_2) = 5\,x_1 + 3\,x_2 + 2\,(x_1\,x_2)^{1/2}$. Then the MRS =

$\dfrac{3 + \left(x_1/x_2\right)^{1/2}}{5 + \left(x_2/x_1\right)^{1/2}}$. This expression cannot be simplified much further.

MRS example 4: Suppose $y = f(x_1,x_2) = 5\,x_1^3\,x_2^2$. Then the MRS = $\dfrac{10\,x_1^3\,x_2}{15\,x_1^2\,x_2^2} = \dfrac{2x_1}{3x_2}$.

MRS example 5: Suppose $y = f(x_1,x_2) = 6\,x_1^{2/3}\,x_2^{1/3}$. Then the MRS =
$\dfrac{2\,x_1^{2/3}\,x_2^{-2/3}}{4\,x_1^{-1/3}\,x_2^{1/3}} = \dfrac{x_1^{2/3}\,x_2^{-2/3}}{2\,x_1^{-1/3}\,x_2^{1/3}}$. But this awkward expression can be further simplified.
Remember that a negative power is just a reciprocal: $x_1^{-1/3} = 1/x_1^{1/3}$ and $x_2^{-1/3} = 1/x_2^{1/3}$. So we can move powers of x_1 or x_2 from numerator to denominator, or vice versa, just by changing the sign of the exponent: MRS = $\dfrac{x_1^{2/3}\,x_1^{1/3}}{2\,x_2^{1/3}\,x_2^{2/3}}$. Remember also that if two

factors have the same base, we multiply them by adding the exponents: MRS = $\frac{x_1^{(2/3)+(1/3)}}{2\,x_2^{(1/3)+(2/3)}} = \frac{x_1}{2\,x_2}$, which is much less awkward than our original answer.

MRS example 6: Suppose $y = f(x_1,x_2) = 8\,x_1^{1/4}\,(x_2-3)^{3/4}$. Then the MRS = $\frac{6x_1^{1/4}(x_2-3)^{-1/4}}{2x_1^{-3/4}(x_2-3)^{3/4}}$. Using the same ideas of negative powers and adding exponents with common bases, this simplifies to MRS $= \frac{3x_1}{(x_2-3)}$.

MRS example 7: Suppose $y = f(x_1,x_2) = 6\,x_1^{1/2} + 4\,x_2^{1/2}$. Then the MRS = $\frac{2\,x_2^{-1/2}}{3\,x_1^{-1/2}} = \frac{2\,x_1^{1/2}}{3\,x_2^{1/2}}$. Again we use the idea that a negative power is just a reciprocal.

MRS example 8: Suppose $y = f(x_1,x_2) = -5\,x_1^{-1} - 3\,x_2^{-1}$. Then the MRS = $\frac{3\,x_2^{-2}}{5\,x_1^{-2}} = \frac{3\,x_1^{2}}{5\,x_2^{2}}$. Again we use the idea that a negative power is just a reciprocal.

MRS example 9: Suppose $y = f(x_1,x_2) = (x_1^{-2} + 3\,x_2^{-2})^{-1/2}$. Then the MRS = $\frac{\left(x_1^{-2}+3x_2^{-2}\right)^{-3/2}3x_2^{-3}}{\left(x_1^{-2}+3x_2^{-2}\right)^{-3/2}x_1^{-3}} = \frac{3x_2^{-3}}{x_1^{-3}} = \frac{3x_1^{3}}{x_2^{3}}$. Again the last step uses the idea that a negative power is just a reciprocal.

The examples above demonstrate that some functions have particularly simple marginal rates of substitution. First, all linear functions (like example 1) have constant marginal rates of substitution. Second, all Cobb-Douglas functions, which are of the form $y = a\,x_1^b\,x_2^c$, have marginal rates of substitution of the form $(c\,x_1)/(b\,x_2)$.

Section 2.10: Summary

Functions of two or more arguments arise frequently in economics. Functions of two arguments can be graphed in two dimensions by plotting all combinations of the arguments that result in a particular value of the function—the results are called level curves. Partial derivatives measure the instantaneous rate of change in the function as one argument is held constant. Linear functions have constant partial derivatives. Partial elasticities measure the same instantaneous rate of change, but in percentage terms. Cobb-Douglas functions have constant partial elasticities. The slopes of level curves, whose absolute values are called marginal rates of substitution, are given by the ratio of the partial derivatives.

Problems

(2.1) [Functions of several variables] Suppose a student's exam score depends on the number of hours the student spent studying and the number of hours the student slept the night before the exam.

a. Write this relationship in function notation, defining all variables.
b. Do the arguments each have positive or negative effects on exam score?
c. What sort of limitation might the student face on possible combinations of the arguments?

(2.2) [Graphing functions of two variables] Suppose $y = x_1 + 2\ x_2$. Assume x_1 and x_2 are nonnegative.

a. On a graph with x_1 on the horizontal axis and y on the vertical axis, plot the function when x_2 is fixed at 2.
b. On a graph with x_1 on the vertical axis and x_2 on the vertical axis, plot the function when y is fixed at 10.

(2.3) [Partial and total derivatives] Suppose y is a function of x_1 and x_2. Suppose at a particular point, the values of the partial derivatives are $\partial y/\partial x_1 = 3$ and $\partial y/\partial x_2 = -2$.

a. If x_1 increases by 0.2 and x_2 remains constant, does y increase or decrease? By approximately how much?
b. If x_2 increases by 0.5 and x_1 remains constant, does y increase or decrease? By approximately how much?
c. If x_1 increases by 0.2 and simultaneously x_2 increases by 0.5, does y increase or decrease? By approximately how much?

(2.4) [Partial and total derivatives] Suppose y is a function of x_1 and x_2. Suppose at a particular point, the values of the partial derivatives are $\partial y/\partial x_1 = 4$ and $\partial y/\partial x_2 = 6$.

a. If x_2 increases by one unit, does y increase or decrease? By approximately how much?
b. Suppose we wish to cancel the effect of this increase in x_2 by changing x_1. That is, we wish to change x_1 so that y returns to its original value. Should we increase or decrease x_1? By approximately how much?

(2.5) [Finding partial derivatives] Find formulas (in terms of x_1 and x_2) for the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for the following functions.

a. $y = 3\ x_1 + 5\ x_2$.
b. $y = 2\ (x_1-5)^{1/2}\ (x_2-4)^{1/2}$.

(2.6) [Finding partial derivatives] Find formulas (in terms of x_1 and x_2) for the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for the following functions.

a. $y = x_1 + 2\ x_2 + 3\ (x_1 x_2)^{1/2}$.
b. $y = 5\ x_1^{1/4}\ x_2^{3/4}$.

(2.7) [Finding partial derivatives] Find formulas (in terms of x_1 and x_2) for the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for the following functions.

a. $y = 7 + 2x_1^{1/2} + 6x_2^{1/2}$.
b. $y = -2/x_1 - 4/x_2$.

(2.8) [Finding partial derivatives] Find formulas (in terms of x_1 and x_2) for the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for the following functions.

a. $y = 2 + 5x_1 + 4x_2$.
b. $y = 8(x_1x_2)^{1/2}$.

(2.9) [Finding partial derivatives] Find formulas (in terms of x_1 and x_2) for the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for the following functions.

a. $y = 3x_1 + 2x_2 + 5(x_1x_2)^{1/2}$.
b. $y = 4\, x_1^3\, x_2^5$.

(2.10) [Finding partial derivatives] Find formulas (in terms of x_1 and x_2) for the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for the following functions.

a. $y = 100 - x_1^{-1/2} - x_2^{-1/2}$.
b. $y = (x_1-5)^3\, x_2^4$.

(2.11) [Approximation rule for products] Total tuition receipts equal tuition per course times total enrollment. Answer the following without using a calculator.

a. If tuition per course increases by 3 percent and total enrollment increases by 2 percent, do total tuition receipts increase or decrease? By approximately what percent?
b. If tuition per course increases by 5 percent and total enrollment decreases by 2 percent, do total tuition receipts increase or decrease? By approximately what percent?

(2.12) [Approximation rule for products] Total cost equals number of units produced times unit cost (also called average cost). Answer the following without using a calculator.

a. If the number of units produced increases by 5 percent and unit cost decreases by 2 percent, does total cost increase or decrease? By approximately what percent?
b. Alternatively, if the number of units produced does not change, but unit cost increases by 3 percent, does total cost increase or decrease? By approximately what percent?

(2.13) [Approximation rule for products] Total revenue received by a seller equals number of units sold times the price. Answer the following without using a calculator.

a. If the number of units sold increases by 3 percent and the price decreases by 1 percent, does total revenue increase or decrease? By approximately what percent?
b. Alternatively, if the number of units sold decreases by 5 percent, but the price increases by 4 percent, does total cost increase or decrease? By approximately what percent?

(2.14) [Approximation rule for ratios] Average hours of work per employee per month equal total hours worked by all employees divided by the number of employees. Answer the following without using a calculator.

a. If total hours worked increase by 5 percent and the number of employees worked increases by 2 percent, do average hours increase or decrease? By approximately what percent?
b. Alternatively, if total hours worked decrease by 3 percent and the number of employees worked decreases by 2 percent, do average hours increase or decrease? By approximately what percent?

(2.15) [Approximation rule for ratios] GDP per capita is defined as GDP divided by population. Answer the following without using a calculator.

a. If GDP increases by 4 percent and the population increases by 1 percent, does GDP per capita increase or decrease? By approximately what percent?
b. Alternatively, if GDP does not change, but the population increases by 2 percent, does GDP per capita increase or decrease? By approximately what percent?

(2.16) [Approximation rule for ratios] Average firm size in an industry is defined as total industry employment divided by the number of firms. Answer the following without using a calculator.

a. If the number of firms decreases by 3 percent and employment in the industry increases by 2 percent, does average firm size increase or decrease? By approximately what percent?
b. Alternatively, if the number of firms decreases by 2 percent and employment in the industry does not change, does average firm size increase or decrease? By approximately what percent?

(2.17) [Meaning of partial elasticity] Suppose the variable y depends on two arguments x_1 and x_2. Suppose the partial elasticity of y with respect to x_1 is $\varepsilon_1 = 3$, and the partial elasticity with respect to x_2 is $\varepsilon_2 = -2$.

a. If x_1 increases by 2 percent and x_2 remains constant, will y increase or decrease? By approximately what percent?
b. If x_1 remains constant and x_2 increases by 5 percent, will y increase or decrease? By approximately what percent?
c. If x_1 increases by 3 percent and simultaneously x_2 increases by 1 percent, will y increase or decrease? By approximately what percent?

(2.18) [Meaning of partial elasticity] For a particular function $y = f(x_1,x_2)$, suppose the partial elasticities sum to one: $\varepsilon_1 + \varepsilon_2 = 1$.

a. If x_1 and x_2 both increase by 2 percent, does y increase or decrease? By approximately what percent?
b. If x_1 and x_2 both increase by 5 percent, does y increase or decrease? By approximately what percent?

(2.19) [Finding partial elasticities] Find formulas for the partial elasticities ε_1 and ε_2 for the following functions.

a. $y = 3\ x_1 + 5\ x_2$.
b. $y = 8\ x_1^{1/4}\ x_2^{3/4}$.

(2.20) [Finding partial elasticities] Find formulas for the partial elasticities ε_1 and ε_2 for the following functions.

a. $y = 4\ x_1^{-1/2}\ x_2^3$.
b. $y = (x_1\text{-}6)^2\ x_2^5$.

(2.21) [Finding partial elasticities] Find formulas for the partial elasticities ε_1 and ε_2 for the following functions.

a. $y = 3\ x_1^2\ (x_2-5)^3$.
b. $y = 7\ x_1^{1/5}\ x_2^{3/5}$.

(2.22) [Finding partial elasticities] Find formulas for the partial elasticities ε_1 and ε_2 for the following functions.

a. $y = 3\ x_1 + 5\ x_2$.
b. $y = 4\ x_1^2\ x_2^{-3}$.

(2.23) [Slope of level curve] Suppose for a particular function $y = f(x_1,x_2)$, the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ are always positive. Now consider the level curves of this function. Must they always slope up or down? Why?

(2.24) [Meaning of MRS] Suppose in the function $y = f(x_1,x_2)$, the marginal rate of substitution of x_2 for x_1 is 3. [Hint: This means that the level curve, drawn with x_1 on the vertical axis and x_2 on the horizontal axis, has a slope of negative 3.]

a. If x_2 increases by 2 units and we wish to keep y constant, must we increase or decrease x_1 ? By approximately how much?
b. If x_2 decreases by ½ units and we wish to keep y constant, must we increase or decrease x_1 ? By approximately how much?

(2.25) [Finding MRS] Suppose $y = f(x_1,x_2) = 3\ x_1 + 5\ x_2$.

a. Find a formula for the partial derivative $\partial y/\partial x_1$.
b. Find a formula for the partial derivative $\partial y/\partial x_2$.
c. Find a formula for the marginal rate of substitution of x_2 for x_1 .

(2.26) [Finding MRS] Suppose $y = f(x_1,x_2) = x_1 + 2\ x_2 + 3\ (x_1x_2)^{1/2}$.

a. Find a formula for the partial derivative $\partial y/\partial x_1$.
b. Find a formula for the partial derivative $\partial y/\partial x_2$.
c. Find a formula for the marginal rate of substitution of x_2 for x_1 .

(2.27) [Finding MRS] Suppose $y = f(x_1,x_2) = 8\, x_1^{1/4}\, x_2^{3/4}$.

a. Find a formula for the partial derivative $\partial y/\partial x_1$.
b. Find a formula for the partial derivative $\partial y/\partial x_2$.
c. Find a formula for the marginal rate of substitution of x_2 for x_1 .

(2.28) [Finding MRS] Suppose $y = f(x_1,x_2) = 2\,(x_1-5)^{1/2}\,(x_2-4)^{1/2}$.

a. Find a formula for the partial derivative $\partial y/\partial x_1$.
b. Find a formula for the partial derivative $\partial y/\partial x_2$.
c. Find a formula for the marginal rate of substitution of x_2 for x_1 .

(2.29) [Finding MRS] Suppose $y = f(x_1,x_2) = -2/x_1 - 4/x_2$.

a. Find a formula for the partial derivative $\partial y/\partial x_1$.
b. Find a formula for the partial derivative $\partial y/\partial x_2$.
c. Find a formula for the marginal rate of substitution of x_2 for x_1 .

(2.30) [Finding MRS] Suppose $y = f(x_1,x_2) = 9\, x_1^{1/3}\, x_2^{2/3}$.

a. Find a formula for the partial derivative $\partial y/\partial x_1$.
b. Find a formula for the partial derivative $\partial y/\partial x_2$.
c. Find a formula for the marginal rate of substitution of x_2 for x_1 .

(2.31) [Finding MRS] Suppose $y = f(x_1,x_2) = 9\, x_1^2\,(x_2-5)$.

a. Find a formula for the partial derivative $\partial y/\partial x_1$.
b. Find a formula for the partial derivative $\partial y/\partial x_2$.
c. Find a formula for the marginal rate of substitution of x_2 for x_1 .

(2.32) [Finding MRS] Suppose $y = f(x_1,x_2) = -(3/x_1) - (4/x_2)$.

a. Find a formula for the partial derivative $\partial y/\partial x_1$.
b. Find a formula for the partial derivative $\partial y/\partial x_2$.
c. Find a formula for the marginal rate of substitution of x_2 for x_1 .

(2.33) [Finding MRS] Suppose $y = f(x_1,x_2) = 6x_1^{1/2} + 4x_2^{1/2}$.

a. Find a formula for the partial derivative $\partial y/\partial x_1$.
b. Find a formula for the partial derivative $\partial y/\partial x_2$.
c. Find a formula for the marginal rate of substitution of x_2 for x_1 .

(2.34) [Functions with identical MRS] Find the MRS of x_2 for x_1 for each of the three functions below. Explain in words why all three functions have the same MRS. [Hint: Refer to the chain rule for finding derivatives of functions of functions.]

a. $y = 5x_1^3 + x_2^2$.
b. $y = (5x_1^3 + x_2^2)^3$.
c. $y = \ln(5x_1^3 + x_2^2)$.

(2.35) [Functions with identical MRS] Find the MRS of x_2 for x_1 for each of the three functions below. Which function has an MRS of x_2 for x_1 which is different from the other two functions' MRSs?

a. $y = 2x_1 + 3x_2$.
b. $y = x_1^2 x_2^3$.
c. $y = x_1^{2/5} x_2^{3/5}$.

(2.36) [Functions with identical MRS] Find the MRS of x_2 for x_1 for each of the three functions below. Which function has an MRS of x_2 for x_1 which is different from the other two functions' MRSs?

a. $y = 5 + \ln(x_1) + \ln(x_2)$.
b. $y = 5 + x_1^{1/2} + x_2^{1/2}$.
c. $y = 5x_1^{1/2} x_2^{1/2}$.

[end of problem set]