LECTURE NOTES ON MICROECONOMICS

ANALYZING MARKETS WITH BASIC CALCULUS

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Part 1: Mathematical tools

Chapter 1: Review of basic calculus

> If in other sciences we should arrive at certainty without doubt and truth without error, it behooves us to place the foundations of knowledge in mathematics.
>
> Roger Bacon (1220-1292)

> Philosophy is written in this grand book—I mean the universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.
>
> Galileo Galilei (1564-1642)

Section 1.1: What is a derivative?

Definition of derivative. Suppose we have a function $y=f(x)$ describing the relationship between variables x and y. For example, $f(x)$ might be a production function, with x denoting the amount of input and y denoting the amount of output. For any two points on this function, we can calculate both the rise (the change in y, or Δy) and the run (the change in x, or Δx). The ratio $\Delta y/\Delta x$ is then the slope of the line connecting those two points (see figure 1.1). It measures the rate at which y responds to x. But unless $y=f(x)$ is a straight line, the slope will vary with the points chosen.

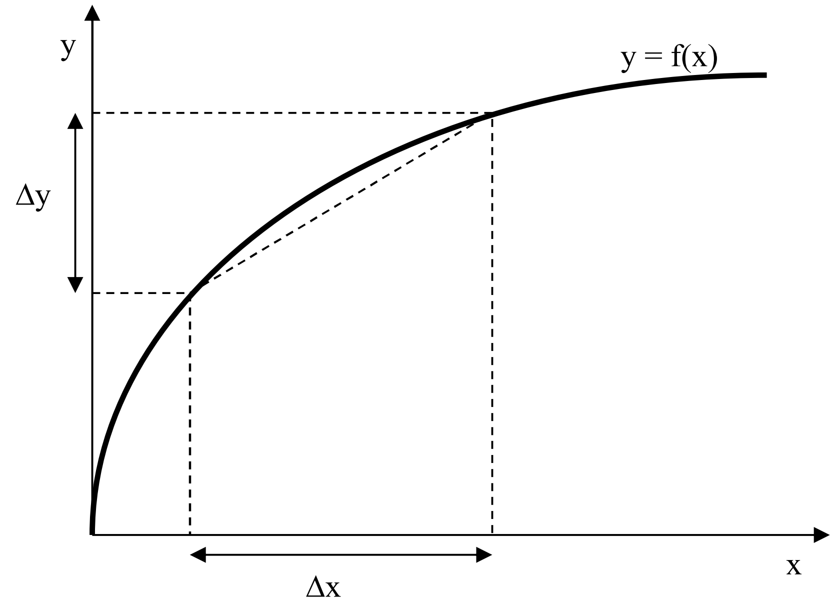
Now suppose we nudge these two points closer together. If $f(x)$ is a smooth curve, the slope will approach a particular value, namely the slope of the line tangent to $f(x)$ at some point (call it x^*) to which our two points are converging. The slope of the line tangent to $f(x)$ at point x^* is called the *derivative* of $f(x)$ at point x^*, or the *derivative of y with respect to x* at point x^*. The formal definition of the derivative is this:

$$(1.1) \qquad \lim_{\Delta x \to 0} \frac{f(x^*+\Delta x) - f(x^*)}{\Delta x}.$$

The sign of the derivative indicates whether the variables are directly or inversely related. If the derivative is positive, the variables are directly (or positively) related. They rise and fall together. If the derivative is negative, the variables are inversely (or negatively) related. They move in opposite directions.

Again, unless $y=f(x)$ is a straight line, the value of the derivative will depend on the point of tangency x^*. So the derivative is itself a function of x. Accordingly, the derivative of the function $f(x)$ is often denoted $f'(x)$. This notation is particularly convenient if we want to express the value of the derivative at a particular point (x). Other ways to denote the derivative function are dy/dx or df/dx. The "d"s recall the "Δ" symbol for change, and serve as a reminder that the derivative is the ratio of infinitesimal changes in f(x) and x.

Figure 1.1. The slope of a function



Meaning of the derivative. The derivative measures the instantaneous rate of change in the function. If x increases by one unit, then y will increase by approximately the value of the derivative. More generally, if x increases by a small amount Δx, then y will increase by approximately Δx times the value of the derivative:

(1.2) $$\Delta y \approx \frac{df}{dx} \Delta x \,.$$

For example, suppose again that x represents input (say, workers) and y represents output of some kind (measured in tons). Suppose at some value of x the derivative $f'(x)=0.03$ tons per worker. Then if the number of workers increases by one, output will increase by approximately $f'(x)$ or 0.03 tons. Similarly, if the number of workers increases by two, output will increase by approximately two times $f'(x)$ or 0.06 tons.

Example: Suppose the derivative of a function $y = f(x)$ equals 2 at a particular point. If x increases by $1/2$, what happens to y? Since $2(1/2) = 1$, y increases by approximately one unit. If x decreases by $3/4$, what happens to y? Since $2(-3/4) = -1.5$, y decreases by approximately 1.5 units.

Example: Suppose the derivative of a function $y = f(x)$ equals -3 at a particular point. If x increases by 2, what happens to y? Since $-3(2) = -6$, y decreases by approximately 6 units. If x decreases by $1/2$, what happens to y? Since $-3(-1/2) = +1.5$, y increases by approximately 1.5 units.

The units of measure of a derivative are the same as the units of measure of a slope, namely the units of y divided by the units of x. In the previous example, the units of measure would therefore be tons per worker. As another example, suppose x represents price in dollars and y represents quantity demanded in gallons. Then the units of measure of the derivative $f'(x)$ are gallons per dollar.

Some functions only rise or only fall. Their derivatives never change sign. The derivative of a *monotonically increasing* function is never negative. The derivative of a *monotonically decreasing* function is never positive.

Section 1.2: Rules for finding derivatives of functions

Formulas for the derivatives of many functions can be found by applying simple rules. Here are some rules useful in microeconomics.

Constant function. Suppose $f(x) = a$, for some fixed constant number a. Then the graph of $f(x)$ is a horizontal line and $df/dx = 0$. For example, if $f(x) = 3$, then $df/dx = 0$.

Power of x. Suppose $f(x) = x^b$, for some fixed number b. Then the derivative of $f(x)$ is $df/dx = b\,x^{b-1}$. For example, if $f(x) = x^3$, then $df/dx = 3x^2$. If $f(x) = x^{1/2}$, then $df/dx = (1/2)\,x^{-1/2}$. Finally, if $f(x) = x$, then $df/dx = 1$.

Negative powers denote reciprocals. For example, $x^{-2} = 1/x^2$ and $x^{-1/2} = 1/x^{1/2}$. However, the same rule for derivatives of powers still applies. So if $f(x) = 1/x^2 = x^{-2}$, then $df/dx = -2\,x^{-3}$. Similarly, if $f(x) = 1/x^{1/2} = x^{-1/2}$, then $df/dx = (-1/2)\,x^{-3/2} = -1/(2x^{3/2})$.

Sum of functions. Suppose $f(x)$ is the sum of two other functions: $f(x) = g(x) + h(x)$. Then the derivative of the sum is the sum of the derivatives: $df/dx = dg/dx + dh/dx$. For example, if $f(x) = x^3 + 2x$, then $df/dx = 3x^2 + 2$.

Product of functions. Suppose $f(x)$ is the product of two other functions: $f(x) = g(x)\cdot h(x)$. Then $df/dx = (dg/dx)\cdot h(x) + g(x)\cdot(dh/dx)$. For example, if $f(x) = x^3\cdot(x + 7)$, then $df/dx = 3x^2\cdot(x + 7) + x^3\cdot 1 = 4x^3 + 21x^2$. Note that functions of the form $f(x) = a\cdot h(x)$, where a is some fixed number, are a special case of this rule; in this case, $df/dx = a\cdot dh/dx$. For example, if $f(x) = 7\,x^2$, then $df/dx = 14x$.

Quotient of functions. Suppose $f(x)$ is the quotient of two other functions: $f(x) = g(x)/h(x)$. Then $df/dx = [(dg/dx)\cdot h(x) - g(x)\cdot(dh/dx)] / [h(x)]^2$. For example, if $f(x) = (2x+4)/x$, then $df/dx = [(2)x - (2x+4)] / x^2 = (-4)/x^2$. The quotient rule is complicated and perhaps difficult to remember. But quotients can always be rewritten as products:

$f(x) = g(x) \cdot [h(x)]^{-1}$, so derivatives of quotients can alternatively be found using a combination of the product rule and the chain rule (see below).

Function of a function. Suppose $f(x)$ is a function of another function: $f(x) = g(h(x))$. Then df/dx is the derivative of the outside function times the derivative of the inside function: $df/dx = (dg/dh)\cdot(dh/dx)$. To see why this works, imagine a production process where x denotes a raw input like coal, h denotes an intermediate product like electricity, and g denotes the final product like steel. Suppose each additional pound of coal yields $dh/dx = 5$ kilowatt-hours of electricity, and each kilowatt-hour of electricity yields $dg/dh = 2$ additional pounds of steel. Then each additional pound of coal yields $df/dx = (dg/dh)\cdot(dh/dx) = 10$ pounds of steel.

Here is an algebraic example. Suppose $f(x) = (3x)^2$. Here, the inside function is $h(x) = (3x)$ and the outside function is $g(h) = h^2$. Applying the rule, $df/dx = (dg/dh)\cdot(dh/dx) = (2h)\cdot(3) = 2(3x)\cdot 3 = 18x$.

Here is another algebraic example. Suppose $f(x) = (x^2+7)^5$. Here, the inside function is $h(x) = (x^2+7)$ and the outside function is $g(h) = h^5$. Applying the rule, $df/dx = (dg/dh)\ (dh/dx) = 5(x^2+7)^4\ (2x) = 10x(x^2+7)^4$.

And another algebraic example. Suppose $f(x) = [2 + (2/x)]^3$. Here, the inside function is $h(x) = [2 + (2/x)]$ and the outside function is $g(h) = h^3$. Applying the rule, $(dg/dh)\ (dh/dx) = df/dx = 3[2 + (2/x)]^2\ (-2/x^2) = -6\ [2 + (2/x)]^2 / x^2$.

And yet another algebraic example. Suppose $f(x) = (6x+7)^{1/2}$. Here, the inside function is $h(x)=6x+7$ and the outside function is $g(h) = h^{1/2}$. Applying the rule, $df/dx = (dg/dh)\ (dh/dx) = (1/2)\ (6x+7)^{-1/2}\ 6 = 3\ (6x+7)^{-1/2} = 3 / (6x+7)^{1/2}$.

This rule for a function of a function is sometimes called the "chain rule" because it can be extended to any number of functions "chained" together: the derivative of a chain of functions is simply the chain of derivatives. Consider a chain of four functions: $f(x) = g(h(k(m(x))))$. The derivative df/dx is simply the chain of derivatives: $df/dx = (dg/dh)\cdot(dh/dk)\cdot(dk/dm)\cdot(dm/dx)$.

Section 1.3: Exponential and natural logarithm functions

Two closely-related functions—the exponential function and the natural logarithm function—deserve special mention because they are extensively used in economics.

Exponential function. The exponential function, denoted $\exp(x)$ or e^x, takes the irrational number $e = 2.718281\ldots$ to the power indicated by the argument x.

This strange-looking function arises naturally in the context of continuously-compounded interest or growth. Suppose you receive 100 percent interest per year. If interest is compounded annually, at the end of one year each dollar will grow to $(1+1.00) = 2$ dollars. If interest is compounded semiannually, at the end of one year each dollar will grow to $\left(1+\frac{1.00}{2}\right)^2 = 2.25$ dollars. If compounded monthly, each dollar will grow to

$\left(1+\frac{1.00}{12}\right)^{12} \approx 2.61$ dollars. More generally, if the year is divided into n subperiods, with compounding at the end of each subperiod, each dollar will grow to $\left(1+\frac{1.00}{n}\right)^{n}$ dollars. The limit of this expression, as n increases without bound, can be shown to equal e = 2.718281… With only slightly more effort, the exponential function can be used to find the result of continuous compounding for any interest rate. For example, if the interest rate is 5 percent, then it can be shown that each dollar will grow to exp(0.05) = $e^{0.05}$ = 1.0513… dollars by the end of one year. More generally, continuous compounding at interest rate r over t years will increase one dollar to exp(rt) = e^{rt} dollars.[1]

Example: Suppose a bank pays 4 percent interest, compounded continuously. Compute the end result of leaving \$10 on deposit for 5 years. Applying the formula just given, the answer is $10\ e^{0.04{\cdot}5} = 10\ e^{0.2} = 10 \times 1.221 = \$12.21$.

Manipulating the exponential function. The exponential function can be manipulated using the usual rules for algebraic manipulation of powers. In particular, exp(x) exp(y) = exp(x+y), exp(x)/exp(y) = exp(x−y), and $[\exp(x)]^y = \exp(xy)$.

Perhaps the most remarkable feature of the exponential function is that *it is its own derivative*: $\frac{d\exp(x)}{dx} = \exp(x)$. In other words, the height of the exponential function (which is necessarily positive) is also its slope. By the chain rule for functions of functions, if f(x) = exp(cx) for some constant number c, then df/dx = c exp(cx). For example, if f(x) = exp(5x), then df/dx = 5 exp(5x).

Example: Find the derivative of $f(x) = \exp(x^2)$. This can be found using the chain rule. Here the inside function is $h(x) = x^2$ and the outside function is g(h) = exp(h). Applying the rule, $df/dx = (dg/dh)\ (dh/dx) = \exp(x^2)\ 2x$.

Example: Find the derivative of h(x) = exp(x+3) . This can also be found using the chain rule. Here the inside function is h(x) = (x+3) and the outside function is g(h) = exp(h). Applying the rule, df/dx = (dg/dh) (dh/dx) = exp(x+3) 1 = exp(x+3).

Natural logarithm function. The natural logarithm, denoted $\log_e(x)$ or ln(x), is defined as the inverse of the exponential function: ln(exp(x)) = x . Thus, ln(x) is the logarithm of x to base e.

The natural logarithm shares the properties of logarithms to any base. First, ln(x) is defined only for positive values of x. Second, ln(x) is greater than zero if x is greater than one, ln(x) is less than zero if x is less than one, and ln(x) equals zero exactly if x equals one. Third, ln(ab) = ln(a) + ln(b). Fourth, $\ln(a^b) = b\ln(a)$.

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[1] This general result can be shown as follows. (The third equal sign exploits the fact that for any number x close to one, $\ln(x) \approx x-1$, shown in the text below.)

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = \lim_{n\to\infty}\exp\left(\ln\left(1+\frac{r}{n}\right)^{nt}\right) = \lim_{n\to\infty}\exp\left(nt\ln\left(1+\frac{r}{n}\right)\right) = \lim_{n\to\infty}\exp\left(nt\left(\frac{r}{n}\right)\right) = \exp(rt).$$

What makes the natural logarithm more convenient than other logarithms is that its derivative is simpler: $\frac{d\ln(x)}{dx} = \frac{1}{x}$.

Example: Find the derivative of $f(x) = \ln(x^3)$. This derivative can be found using the chain rule. Here the inside function is $h(x) = x^3$ and the outside function is $g(h) = \ln(h)$. Applying the rule, $df/dx = (dg/dh)(dh/dx) = (1/x^3)\, 3x^2 = 3/x$.

Example: Find the derivative of $f(x) = \ln(2x+3)$. This derivative also can be found using the chain rule. Here the inside function is $h(x) = (2x+3)$ and the outside function is $g(h) = \ln(h)$. Applying the rule, $df/dx = (dg/dh)(dh/dx) = [1/(2x+3)]\, 2 = 2/(2x+3)$.

Approximating logarithms. Applying the general approximation formula (1.2) we find the following convenient formula:

$$(1.3) \qquad \Delta \ln(x) \approx \frac{1}{x}\Delta x = \frac{\Delta x}{x} .$$

In words, *the absolute change in the natural logarithm of x equals approximately the percent change in x itself.* For example, if x increases by 2 percent, then its natural logarithm increases by about 0.02. If x decreases by 3 percent, then its natural logarithm decreases by about 0.03. (Check this on a calculator!)[2]

This approximation formula, in conjunction with the fact that $\ln(1) = 0$, implies that for any number x close to one, $\ln(x) \approx x - 1$. For example, $\ln(1.02)$ equals approximately 0.02, and $\ln(0.97)$ equals approximately -0.03.

Example: Suppose $y = \ln(x)$. If x increases by 4 percent, what happens to y? Using formula (1.3), ln(x) increases by about 0.04.

Example: Estimate the following without using a calculator: ln(1.01), ln(0.98), ln(1.025). Using the approximation formula just given, $\ln(1.01) \approx 1.01-1 = 0.01$, $\ln(0.98) \approx 0.98-1 = -0.02$, and $\ln(1.025) \approx 1.025-1 = 0.025$.

Section 1.4 Optimization

Optimization in economics. A fundamental assumption in economics is that economic agents (consumers, workers, firms) *optimize*, or do the best they can with what they have. Optimization might mean maximization (choosing the largest) or minimization (choosing the smallest), depending on context. Obviously, most economic agents do not use mathematical techniques to make economic decisions. Rather, they use intuition or trial-and-error. They may never find the exact optimum for any particular situation. But they are probably roughly correct on average, so if we find the exact optimum with mathematical techniques we will have a reasonable prediction of average behavior.

[2] The approximation formula is less accurate for double-digit percent changes. For example, if x increases by 20 percent, then its natural logarithm increases by 0.182. If x increases by 30 percent, then its natural logarithm increases by 0.262.

The assumption that economic agents optimize has proved fruitful for at least two reasons. First, it is *precise*. It yields specific predictions, provided an agent's objectives and available choices are specified. Second, it is *tractable*. Many mathematical techniques have been developed to solve optimization problems. Here, we review techniques that use derivatives.

The first-order necessary condition. Suppose the function $y = f(x)$ represents the relationship of output x to a firm's profits y. Assume the firm wishes to set output to maximize profits. Thus x is the firm's *decision variable* or *control variable* and $f(x)$ is the firm's *objective function*. How can we describe the firm's profit-maximizing choice x^* ?

Consider the derivative of $f(x)$. In particular, consider the value of the derivative at some proposed choice x^*. If the value of this derivative $f'(x^*)$ were positive, then the firm could increase profits by increasing x , for then profits would rise by approximately the value of the derivative times the change in x . On the other hand, if the value of this derivative were negative, then the firm could increase profits by decreasing x , for then profits would rise by approximately the (negative) value of the derivative times the (negative) change in x . It follows that, at the profit-maximizing choice x^*, the derivative can be neither positive nor negative. It must be zero. This condition can be written as

(1.4) $$f'(x^*) = 0 \quad \text{or} \quad \left.\frac{df}{dx}\right|_{x=x^*} = 0$$

It is called the *first-order necessary condition* (FONC) because it concerns the first derivative of $f(x)$ and, by the argument just given, it is clearly necessary for an optimum (with exceptions to be noted below.)

An analogous argument holds for minimization. Suppose $y = f(x)$ represents the relationship of the firm's unit cost y to its degree of automation x. If the value of the derivative were positive, the firm could decrease cost by decreasing x . If the value of the derivative were negative, the firms could decrease cost by increasing x. It follows that, at the optimal choice x^*, the derivative can be neither positive nor negative. Thus the FONC is the same for miminization as for maximization. If the equation expressing the FONC can be solved, we can find the optimal choice x^*.

Example: Suppose we wish to choose x to maximize $f(x) = 6x - x^2$. Then the FONC is $0 = 6 - 2x$, so $x^* = 3$.

Example: Suppose $f(x) = 20 \ln(x) - 2x$. Then the FONC is $0 = 20/x - 2$, so $x^* = 10$.

Qualifications and exceptions. If a function has both peaks and valleys, then the FONC will characterize all of them. There will be multiple solutions to the FONC, not all of them interesting. If the function can be easily graphed, peaks and valleys may be easily distinguished. If graphing is difficult, then we can distinguish peaks from valleys by examining how the derivative changes in the vicinity of the proposed solution x^*. If x^* represents a peak, then the derivative will be positive for $x<x^*$ and negative for $x>x^*$ (see figure 1.2). Thus in the vicinity of x^*, the derivative decreases as x increases. A decreasing function has a negative derivative, so we could say that the derivative of the

Figure 1.2. The maximum of a function

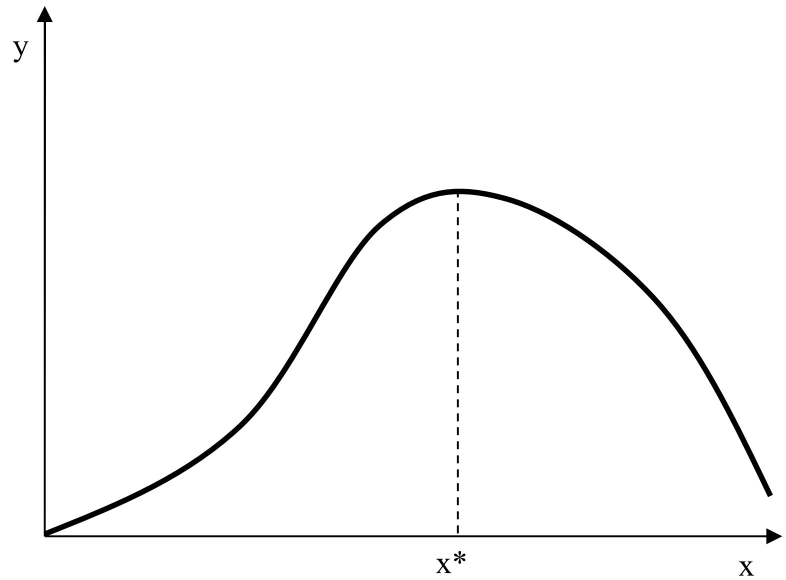
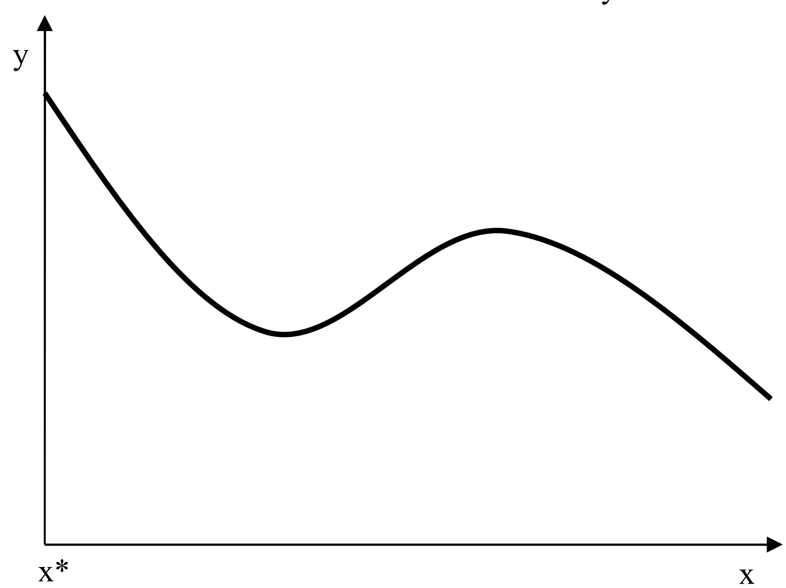


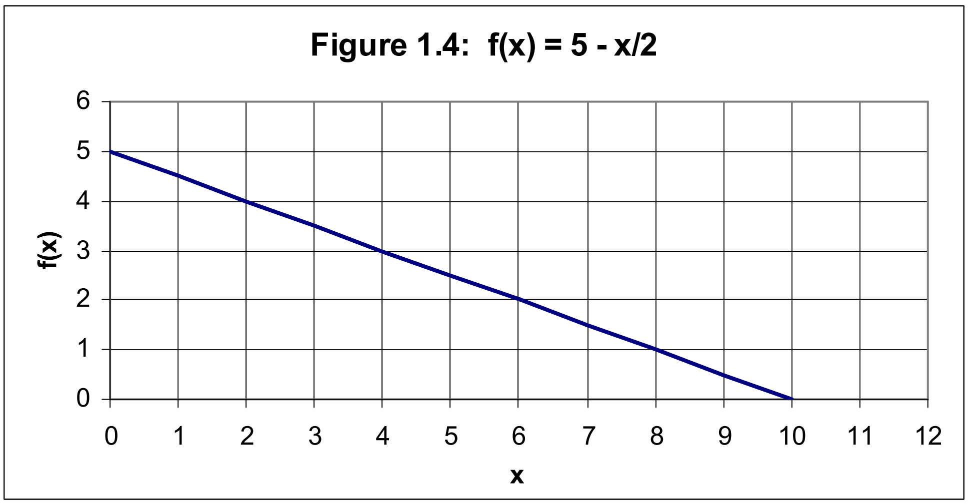
Figure 1.3. A function whose maximum is at a boundary



derivative—or second derivative—is negative. This is the so-called second-order condition. In all examples and problems in these notes, the second-order condition will be satisfied so it will not be necessary to check it.

However, another issue will often need checking. Many economic variables cannot meaningfully be negative. For example, a firm cannot choose a negative level of output. If x cannot be negative, then the maximum value of the function $f(x)$ could occur at $x^*=0$ without $f'(0)=0$ (see figure 1.3). More generally, whenever the domain of x is bounded, we must check for an optimum at that boundary, not just where the derivative is zero.

Example: Suppose $f(x) = 5 - x/2$, but x is required to be nonnegative. The derivative df/dx is never zero, but the maximum value of the function occurs at $x^* = 0$ and $f(x^*) = 5$ (see figure 1.4)



Example: A more complicated example is maximizing $f(x) = -x^3 + 15x^2 - 72x + 50$, where again x is required to be nonnegative. It is easy to show that the derivative df/dx equals zero when $x = 4$ or $x = 6$. But these values do not maximize the function, for $f(4) = -62$ and $f(6) = -58$, whereas $f(0) = 50$. Note that in this example and the last, the derivative is negative at the boundary $x=0$. This shows that the function falls as x moves away from the boundary, as we would expect if the maximum occurs at the boundary (see figure 1.5).

Section 1.5 What is an elasticity?

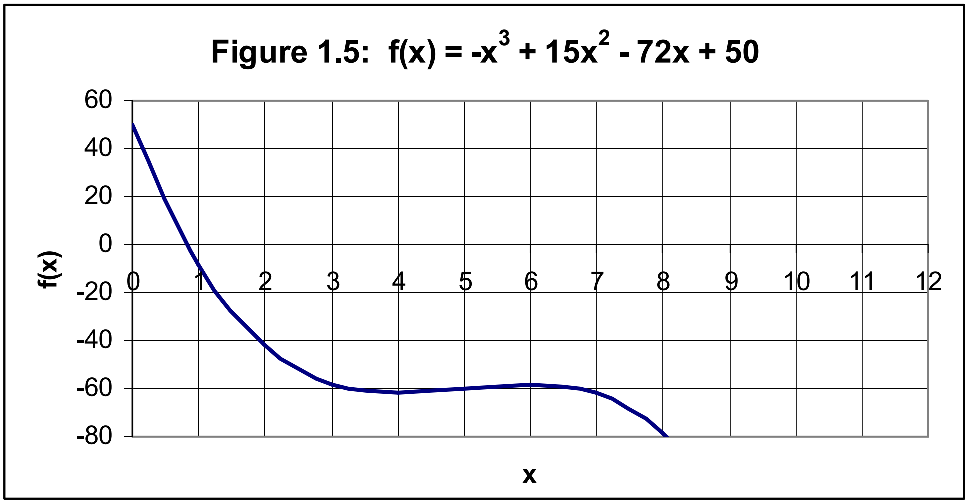
Definition of elasticity. Suppose we have a function $y=f(x)$ describing the relationship between variables x and y . For any two points on this function, we can calculate both the percent change in y (or $\Delta y/y$) and the percent change in x (or $\Delta x/x$). The ratio of these changes,

$$\frac{\Delta y / y}{\Delta x / x} = \left(\frac{\Delta y}{\Delta x}\right) \cdot \left(\frac{x}{y}\right), \tag{1.5}$$

is an alternative measure of the rate at which y responds to x.

Suppose we nudge these two points closer together. If f(x) is a smooth curve, the ratio of $(\Delta y/\Delta x)$ will approach the derivative, and the entire ratio of percent changes will approach a particular value , called the *elasticity of y with respect to x.* Thus the formal definition of the elasticity is this:

$$(1.6) \qquad \varepsilon = \left(\frac{dy}{dx}\right)\cdot\left(\frac{x}{y}\right) = f'(x)\cdot\left(\frac{x}{y}\right).$$



Elasticities are normally calculated only for variables that are strictly positive, so the sign of the elasticity must be the same as the sign of the derivative. The elasticity is positive for variables that are directly or positively related. The elasticity is negative for variables that are inversely or negatively related.

For most functions, the value of the elasticity will vary with x, so the elasticity is itself a function of x. Some interesting exceptions will be discussed below.

Meaning of the elasticity. The elasticity measures the instantaneous rate of change in the function in percentage terms. If x increases by one percent, then y will increase in percent by approximately the value of the elasticity. More generally, if x increases by a small percent $(\Delta x/x)$, then y will increase by a percent approximately equal to the value of the elasticity times $(\Delta x/x)$:

$$(1.7) \qquad \frac{\Delta y}{y} \approx \varepsilon\,\frac{\Delta x}{x}.$$

(This formula follows from approximation formula (1.2) and the elasticity definition (1.6).) For example, suppose again that x represents workers and y represents output, and the elasticity value is 0.75. If the number of workers increases by 1%, then output will increase by approximately 0.75%. Similarly, if the number of workers increases by 2%, then output will increase by approximately 1.5%.

Example: Suppose the elasticity of a function $y = f(x)$ equals $1/2$ at a particular point. If x increases by 4 percent, what will happen to y? Applying formula (1.7), the percent change in y is approximately $(1/2)\cdot 4$ percent = 2 percent. So y will increase by approximately 2 percent. If x decreases by 6 percent, what will happen to y? Again applying (1.7), the percent change in y is approximately $(1/2)\cdot(-6)$ percent = -3 percent. So y will decrease by approximately 3 percent.

No units of measure. Because it is a ratio of percent changes, an elasticity has no units of measure. A percent change in price is the same whether the price is measured in dollars or cents. A percent change in quantity is the same whether the quantity is measured in gallons or liters. Unlike a derivative, therefore, an elasticity is a pure number. This convenient feature of elasticity has encouraged its use in place of slope to measure the strength of all sorts of economic relationships, such as the elasticity of demand with respect to price or income, the elasticity of supply with respect to price, and so forth.

Elasticities and natural logarithms. An alternate definition of elasticity can be given in terms of the logarithms of the variables x and y. Recall that $d\ln(x)/dx = 1/x$. Similarly, $d\ln(y)/dy = 1/y$. In words, "the percent change in a variable is approximately equal to the absolute change in its logarithm." Substitute these results into the definition of elasticity to get the following.

$$(1.8) \qquad \varepsilon = \left(\frac{dy}{dx}\right)\cdot\left(\frac{x}{y}\right) = \frac{dy\cdot\left(\frac{1}{y}\right)}{dx\cdot\left(\frac{1}{x}\right)} = \frac{dy\cdot\left(\frac{d\ln(y)}{dy}\right)}{dx\cdot\left(\frac{d\ln(x)}{dx}\right)} = \frac{d\ln(y)}{d\ln(x)}.$$

Thus elasticity can be defined as $d\ln(y)/d\ln(x)$. This alternative definition is particularly useful if the function itself can be more easily expressed as a relationship between logarithms. For example, if $\ln(y) = 5 - 0.3\ln(x)$, then the elasticity is -0.3.

Section 1.6 Finding elasticities of functions

General approach. Using the formal definition of elasticity given in equation (1.6), formulas for the elasticity can easily be found. For example, if $y = 200 - 50x$, then $\varepsilon = (-50)(x/y) = (-50x)/(200-50x)$. If $y = 1 - (2/x)$, then $\varepsilon = (2/x^2)(x/y) = 2/(x-2)$. Finally, if $y = 3x^{-1/2}$, then $\varepsilon = (-3/2)x^{-3/2}(x/y) = -1/2$.

Powers of x. The last example is a case where the elasticity is a constant, not a function of x. In fact, all functions that are powers of x have a constant elasticity equal to that power. For suppose $y = a x^b$ for fixed constants a and b. Then using the formal definition of elasticity, we can derive the following.

$$(1.9) \qquad \varepsilon = \left(\frac{dy}{dx}\right)\cdot\left(\frac{x}{y}\right) = \left(abx^{b-1}\right)\left(\frac{x}{ax^b}\right) = b$$

Example: Which of the following functions has a constant elasticity? f(x) = 2 + x, g(x) = x^3, and h(x) = 5x. Here, f(x) is not a power function so it does not have a constant elasticity. But g(x) is a power function and its elasticity is 3. Also, h(x) is a power function and its elasticity is 1.

Section 1.7 Computing elasticities from datapoints

Suppose we have two datapoints on x and y. For example, suppose we have the points (x,y) = (3,4), (27,16) How can we estimate the elasticity of y with respect to x? One is tempted to simply apply equation (1.5). For this example, Δx = 27–3 = 24 and Δy = 16–4 = 12. But what values should we use for x and y in equation (1.5)? If we choose (x,y) = (3,4), then ε = (12/24) (3/4) = 3/8 = 0.375. If we choose (x,y) = (27,16), then ε = (12/24) (27/16) = 27/32 = 0.84375. These answers are very different. We need a way to resolve the ambiguity in choice of values for x and y.

Arc elasticity formula: One commonly-used approach is to choose x and y to be the average or *midpoint* of the two datapoints:

$$\varepsilon = \left(\frac{\Delta y}{\Delta x}\right) \cdot \left(\frac{(x_1 + x_2)/2}{(y_1 + y_2)/2}\right) . \tag{1.10}$$

In this example, we would choose x to be (3+27)/2 = 15 and choose y to be (4+16)/2 = 10. Then ε = (12/24) (15/10) = 3/4 = 0.75. Note that this answer lies between the two answers given above.

Difference in logarithms formula: Another commonly-used approach applies equation (1.8). It computes the elasticity as the ratio of the difference in the logarithms.

$$\varepsilon = \left(\frac{\Delta \ln(y)}{\Delta \ln(x)}\right) . \tag{1.11}$$

In this example, Δln(x) = ln(27) – ln(3) = 3.2958 − 1.0986 = 2.1972. Similarly, Δln(y) = ln(16) – ln(4) = 2.7726 − 1.3863 = 1.3863. Therefore ε = 0.63. This answer also lies between the answers given above.

Which formula is better—the arc elasticity or the difference in logarithms? There is no general answer. However, if the relationship between x and y has a constant elasticity, the difference in logarithms will yield that elasticity exactly.

Section 1.8 Summary

Derivatives measure the instantaneous rate of change along a function as the ratio of small changes in y to small changes in x. Straight lines have constant derivatives. Formulas for the derivatives of many functions can be found using a few simple rules. The maximum or minimum value of a function can be found by setting the derivative of the function equal to zero, subject to some qualifications and exceptions. Elasticities also measure instantaneous rate of change, but as the ratio of small percent changes. Powers of x have constant elasticities.

Problems

(1.1) [Definition of derivative] Plot the function $f(x) = 2 + (1/2)\,x$ on graph paper, for $x = 0$ through 4. Is the derivative of this function constant or changing? Estimate its value from the graph. Is this a monotonic function?

(1.2) [Definition of derivative] Plot the function $f(x) = 1 + (1/2)\,x^2$ on graph paper, for $x = -1$ through 3. Is the derivative of this function constant or changing? Estimate its value from the graph at $x = 0$ and again at $x = 1$. Is this a monotonic function?

(1.3) [Derivatives—numerical approximation] Suppose $y = f(x) = x^2$ and consider the point $x^*=1$.

a. Compute the slope of this function between $x^*=1$ and $x^{**}=2$. That is, compute $\dfrac{f(2)-f(1)}{2-1}$.
b. Similarly, compute the slope of this function between $x^*=1$ and $x^{**}=1.5$.
c. Similarly, compute the slope of this function between $x^*=1$ and $x^{**}=1.25$.
d. Find the formula for the derivative of $f(x)$ and evaluate it at $x^*=1$.

(1.4) [Using derivatives] Suppose the derivative of the function $y = f(x)$ is 5.

a. If x increases by 2 units, does y increase or decrease? By approximately how much?
b. If x decreases by 0.4 units, does y increase or decrease? By approximately how much?

(1.5) [Using derivatives] Suppose the derivative of the function $y = f(x)$ is -0.3.

a. If x increases by 0.7 units, does y increase or decrease? By approximately how much?
b. If x decreases by 0.5 units, does y increase or decrease? By approximately how much?

(1.6) [Finding derivatives] Find formulas for the derivatives with respect to x of the following functions.

a. $y = 6\,x^{1/3}$.
b. $y = -5x^2 + 10x + 5$.
c. $y = (x-5)^2$.

(1.7) [Finding derivatives] Find formulas for the derivatives with respect to x of the following functions.

a. $y = 3/x$.
b. $y = 4\,x^{-5}$.
c. $y = (x+2)^{1/2}$.

(1.8) [Finding derivatives] Find formulas for the derivatives with respect to x of the following functions.

a. $y = 3x^{-3} + (1/x)$.
b. $y = 5x^{1.5}$.
c. $y = 1 / (x-3)$.

(1.9) [Finding derivatives] Find formulas for the derivatives with respect to x of the following functions.
a. $y = 3x^2 + 5x + 2$
b. $y = (x+1)^2$
c. $y = 5x^3 (4 + x)$

(1.10) [Finding derivatives] Find formulas for the derivatives with respect to x of the following functions.
a. $y = (1/2) x^2 + 5 x^{1/2} + 7$
b. $y = (5x + 2)^{1/2}$
c. $y = 5x^2 (x+4)$

(1.11) Find derivatives of the following functions:
a. $y = 0.2 x^3 - 0.4 x^2 + 2.5 x + (10/x)$.
b. $y = (x^2 + 2x)^5$.
c. $y = (3x+5)/x$.

(1.12) [Using exponential function] Suppose $100 is deposited in a bank that pays 3 percent interest, compounded continuously. Use a calculator with an exponential key to calculate the amount in this account after 1 year, 5 years, and 10 years.

(1.13) [Derivatives of exponential function] Find formulas for the derivatives with respect to x of the following functions.
a. $y = \exp(3x)$.
b. $y = \exp(5 + 2x^2)$.
c. $y = \exp(\exp(\exp(x)))$.

(1.14) [Natural logarithm] Prove that $\exp[\ln(x)+\ln(y)] = xy$.

(1.15) [Using natural logarithm function] Suppose $y = \ln(x)$, where $\ln(.)$ denotes the natural logarithm function. Answer the following questions without using a calculator.
a. If x increases by 1.5 percent, does y increase or decrease? By how much, approximately?
b. If x decreases by 3.5 percent, does y increase or decrease? By how much, approximately?

(1.16) [Derivatives of natural logarithm function] Find formulas for the derivatives with respect to x of the following functions. . [Note: ln(.) denotes the natural logarithm function.]

a. $y = \ln(7x)$.
b. $y = 3 \ln(5x^2)$.
c. $y = 3 x^{-3} + \ln(x)$.

(1.17) [Exponential and natural logarithm functions] Simplify the following expressions. [Note: ln(.) denotes the natural logarithm function.]

a. $\ln [\exp(x)]$.
b. $\ln[\exp(x) \exp(y)]$.
c. $\exp[\ln(x) + \ln(y)]$.

(1.18) [Natural logarithm function] Estimate the following without using a calculator. [Note: ln(.) denotes the natural logarithm function.]

a. $\ln(1.015)$.
b. $\ln(0.982)$.
c. $\ln(1.022)$.

(1.19) [Maximizing a function] Maximize the function $y = f(x) = -5x^2 + 30x + 3$. In particular, find the maximizing value of the argument: x^*. Then calculate the maximum value of the function: $y^*=f(x^*)$. For what range of values of x does this function slope up? For what range of values does it slope down?

(1.20) [Maximizing a function] Maximize the function $y = f(x) = \ln(x) - (x/5)$. In particular, find the maximizing value of the argument: x^*. Then calculate the maximum value of the function: $y^*=f(x^*)$. For what range of values of x does this function slope up? For what range of values does it slope down?

(1.21) [Minimizing a function] Minimize the function $y = f(x) = 2x - 5 \ln(x)$. In particular, find the minimizing value of the argument: x^*. Then calculate the minimum value of the function: $y^*=f(x^*)$. For what range of values of x does this function slope up? For what range of values does it slope down?

(1.22) [Maximizing a function] Maximize the function $y = f(x) = -3 - 2x$, subject to the restriction that $x \geq 0$. In particular, find the maximizing value of the argument: x^*. Then calculate the maximum value of the function: $y^*=f(x^*)$. For what range of values of x does this function slope up? For what range of values does it slope down? [Hint: Graph the function first.]

(1.23) [Maximizing a function] Maximize the function $y = f(x) = -x^2 - 6x + 10$, subject to the restriction that $x \geq 0$. In particular, find the maximizing value of the argument: x^*. Then calculate the maximum value of the function: $y^*=f(x^*)$. For what range of values of x does this function slope up? For what range of values does it slope down? [Hint: Graph the function first.]

(1.24) [Definition of elasticity] True or false? If a function slopes down, its elasticity must be negative. Justify your answer.

(1.25) [Using elasticities] Suppose the elasticity of the function $y = f(x)$ is 3.

a. If x increases by 3 percent, does y increase or decrease? By approximately how much?
b. If x decreases by 2 percent, does y increase or decrease? By approximately how much?

(1.26) [Using elasticities] Suppose the elasticity of the function $y = f(x)$ is $-1/2$.

a. If x increases by 2 percent, does y increase or decrease? By approximately how much?
b. If x decreases by 4 percent, does y increase or decrease? By approximately how much?

(1.27) [Using elasticities] Suppose the elasticity of the function $y = f(x)$ is 0.25.

a. If x increases by 4 percent, does y increase or decrease? By approximately how much?
b. If x decreases by 2 percent, does y increase or decrease? By approximately how much?

(1.28) [Using elasticities] Suppose the elasticity of the function $y = f(x)$ is -1.5.

a. If x increases by 2 percent, does y increase or decrease? By approximately how much?
b. If x decreases by 3 percent, does y increase or decrease? By approximately how much?

(1.29) [Elasticities and natural logarithms] Find formulas in terms of x alone for the elasticity of y with respect to x for the following functions.

a. $\ln(y) = 3 + 2 \ln(x)$
b. $\ln(y) = 5 - 3 \ln(x)$
c. $\ln(y) = 7 + (\ln(x)/2)$.

(1.30) [Finding elasticities] Find formulas in terms of x alone for the elasticity of y with respect to x for the following functions.

a. $y = 20 - 4x$.
b. $y = 3 x^2$.
c $y = 2x$.

(1.31) [Finding elasticities] Find formulas in terms of x alone for the elasticity of y with respect to x for the following functions.

a. $y = 10 - x$.
b. $y = 3/x$.
c $y = 5 x^{0.3}$.

(1.32) [Finding elasticities] Find formulas in terms of x alone for the elasticity of y with respect to x for the following functions.

a. $y = 15 - 2x$.
b. $y = x^{1/2}$.
c $y = 2 / x^3$.

(1.33) [Finding elasticities] Find formulas in terms of x alone for the elasticity of y with respect to x for the following functions.

a. $y = 5$

b. $y = 5 - 2x$

c. $y = x^2 + 5 x$.

(1.34) [Finding elasticities] Find formulas in terms of x alone for the elasticity of y with respect to x for the following functions.

a. $y = 3x^2$
b. $y = 15x^{-2}$,
c. $y = 7x^{1/3}$.

(1.35) [Functions with constant elasticity or slope] Suppose the elasticity of y with respect to x is -2 when $x = 10$ and $y = 30$.

a. First, suppose that the *elasticity* of the relationship is constant for all x. Find the equation that relates y to x. [Hint: Constant elasticity implies $y = ax^b$. Now find a and b.]
b. Alternatively, suppose the *slope* of the relationship is constant for all x. Find the equation that relates y to x. [Hint: Constant slope implies $y = a + b x$. Now find a and b.]

(1.36) [Computing elasticity: arc formula] Compute the elasticity of y with respect to x between the following pairs of points using the arc elasticity (or midpoint) formula. Be sure the sign of the elasticity is correct.

a. (x,y) = (4,5) and (8,15).
b. (x,y) = (2,9) and (14,15).
c. (x,y) = (6,8) and (4, 12).

(1.37) [Computing elasticity: arc formula] Compute the elasticity of y with respect to x between the following pairs of points using the arc elasticity (or midpoint) formula. Be sure the sign of the elasticity is correct.

a. (x,y) = (5,4) and (7,2).
b. (x,y) = (8,10) and (4,14).
c. (x,y) = (10,7) and (2,5).

(1.38) [Computing elasticity: difference in logs] Compute the elasticity of y with respect to x between the following pairs of points using the difference-in-logs formula. Give your answer to four significant digits. Be sure the sign of the elasticity is correct.

a. (x,y) = (4,5) and (8,15).
b. (x,y) = (2,9) and (14,15).
c. (x,y) = (6,8) and (4, 12).

(1.39) [Computing elasticity: difference in logs] Compute the elasticity of y with respect to x between the following pairs of points using the difference-in-logs formula. Give your answer to four significant digits. Be sure the sign of the elasticity is correct.

a. (x,y) = (5,4) and (7,2).
b. (x,y) = (8,10) and (4,14).
c. (x,y) = (10,7) and (2,5).

[end of problem set]